# Certain Properties of the j-generalized p - k Mittag-Leffler Function

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## Abstract

The aim of this paper is to study various properties of j-generalized p - k Mittag-Leffler function. Mellin-Barnes integral representation of j-generalized p-k Mittag-Leffler Function have been derived. The relationships of j-generalized p-k Mittag-Leffler Function with Fox H-Function and Wright hypergeometric function also been established.Few well known Integral transforms viz. Euler transform, Laplace Transform and Mellin transform also obtained. Finally we derive some particular cases.

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## 1 Introduction

### 1.1 Definition

Let $x \in \mathbb{C}; k, p \in \mathbb{R}^+ \setminus \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ defined as

$$ {}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p).........(\frac{xp}{k} + (n-1)p). \tag{1.1} $$

and two parameters gamma function defined as [9]

## 1.2 Definition

For $x \in \mathbb{C} \setminus k\mathbb{Z}^-; k, p \in \mathbb{R}^+ \setminus \{0\}$ and $Re(x) > 0, n \in \mathbb{N}$, the p - k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k(x)$ as

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}} \tag{1.2} $$

or

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}} \tag{1.3} $$

The integral representation of p - k Gamma Function is given by

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt \tag{1.4} $$

$$ {}_p\Gamma_k(x) = (\frac{p}{k})^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma(\frac{x}{k}) \tag{1.5} $$

$$ {}_p(x)_{n,k} = (\frac{p}{k})^n (x)_{n,k} = (p)^n (\frac{x}{k})_n \tag{1.6} $$

Also for Generalized p - k Pochhammer Symbol, we have

$$ {}_p(x)_{nq,k} = (\frac{p}{k})^{nq} \, (x)_{nq,k} = (p)^{nq} \, (\frac{x}{k})_{nq} = (pq)^{nq} \prod_{r=1}^{q} (\frac{\frac{x}{k} + r - 1}{q})_n \tag{1.7} $$

$$ {}_p(x)_{n,k} = \frac{{}_p\Gamma_k(x + nk)}{{}_p\Gamma_k(x)} \tag{1.8} $$

$$ {}_p\Gamma_k(x + k) = \frac{xp}{k} \, {}_p\Gamma_k(x) \tag{1.9} $$

$$ np \, {}_p(x)_{n-1,k} = {}_p(x)_{n,k} - {}_p(x - k)_{n,k} \tag{1.10} $$

And

$$ {}_p(x)_{n+j,k} = {}_p(x)_{j,k} \times {}_p(x + jk)_{n,k} \tag{1.11} $$

Gehlot and Bhandari [10], Introduced the j-generalized p- k Mittag-Leffler function, denoted by ${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z)$ and defined for $k, p \in \mathbb{R}^+ \setminus \{0\}; \alpha, \beta, \gamma \in \mathbb{C}/k\mathbb{Z}^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$, $j \in \mathbb{N}_0$ and $q > 0$.
The j-generelized p - k Mittag-Leffler function denoted by ${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z)$ and defined as

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\mathit{\Gamma}_k(n\alpha + \beta)} \frac{z^n}{(n+j)!}, \qquad z \in \mathbb{C}. \tag{1.12} $$

Where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1.1) and ${}_p\mathit{\Gamma}_k(x)$ is the two parameter Gamma function given by equation (1.3).

Wright generalized hypergeometric function [15];

$$ {}_p\psi_q \left[ \begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ \qquad\qquad\qquad z \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \mathit{\Gamma}(\alpha_i + A_i n) z^n}{\prod_{j=1}^{q} \mathit{\Gamma}(\beta_i + B_i n) n!} \tag{1.13} $$

$$ {}_p\psi_q \left[ \begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ \qquad\qquad\qquad z \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{array} \right] = H_{p,q+1}^{1,p} \left[ -z \mid \begin{array}{c} (1-\alpha_1, A_1), \ldots, (1-\alpha_p, A_p); \\ \\ (0,1), (1-\beta_1, B_1), \ldots, (1-\beta_q, B_q); \end{array} \right] \tag{1.14} $$

Where $H^{m,n}_{p,q}[.]$ denotes the Fox H-function.
Euler Beta transform,[6],

$$B[f(z):a,b]=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz \tag{1.15}$$

Laplace transform,([7],equation3.1.1),

$$L[f(z):s]=\int_0^\infty e^{-sz}f(z)dz \tag{1.16}$$

Mellin transform,([7],equation 4.1.1),

$$M[f(z):s]=\int_0^\infty z^{s-1}f(z)dz=f^*(s), Re(s)>0 \tag{1.17}$$

then

$$f(z)=M^{-1}[f^*(s):x]=\int f^*(s)x^{-s}ds \tag{1.18}$$

Notations: Let $\mathbb{C},\mathbb{R}^+=(0,\infty),Re(),\mathbb{Z}^-,\mathbb{N}_0$ and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, real part of complex number, negative integer, whole number and natural numbers respectively.

## 2 Recurrence Relation and Integral Representation of the j-generalized p - k Mittag-Leffler Function

In this section we evaluate the recurrence relations and integral representations of the j-generalized p - k Mittag-Leffler function.

**Theorem 2.1** For $k,p\in\mathbb{R}^+\setminus\{0\}$; $\alpha+r,\beta+s+k,\gamma\in\mathbb{C}\setminus k\mathbb{Z}^-$; $R(\alpha+r)>0,R(\beta+s+k)>0,R(\gamma)>0,q>0,j\in\mathbb{N}_0$, we get

$$\begin{aligned}{}^j_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z)-p\,{}^j_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z)=\frac{p^2}{k^2}\Big[(\alpha+r)^2z^2\,{}^j_p\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)\\+\Big\{(\alpha+r)^2+(\alpha+r)(2\beta+2s+2k)z\Big\}\,{}^j_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)\\+(\beta+s)(\beta+s+2k)\,{}^j_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)\Big]\end{aligned} \tag{2.1}$$

where ${}^j_p\dot{E}^{\gamma,q}_{k,\alpha,\beta}(z)=\frac{d}{dz}\,{}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and ${}^j_p\ddot{E}^{\gamma,q}_{k,\alpha,\beta}(z)=\frac{d^2}{dz^2}\,{}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z)$.

**Proof:** The j-generalized p-k Mittag-Leffler function, from equation (1.12),

$${}^j_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+k)((n+j)!)}$$

using equation (1.9), we have

$${}^j_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z)=\sum_{n=0}^{\infty}\frac{k}{p}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s\}((n+j)!)}, \tag{2.2}$$

and

$${}^j_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+2k)((n+j)!)} \tag{2.3}$$

using equation (1.9), we can write

$$ {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

$$ \times\frac{k^2}{p^2}\frac{1}{\{n(\alpha+r)+\beta+s)\}\{n(\alpha+r)+\beta+s+k)\}} $$

i.e.

$$ {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z)=\sum_{n=0}^{\infty}\frac{k}{p^2}[\frac{1}{(n(\alpha+r)+\beta+s)}-\frac{1}{(n(\alpha+r)+\beta+s+k)}] $$

$$ \times\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

This can be written as

$$ {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z)=\frac{k}{p^2}[\frac{p}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z)-S] $$

where

$$ S=\frac{p}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z)-\frac{p^2}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z) \tag{2.4} $$

$$ S=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k)\}((n+j)!)} \tag{2.5} $$

applying the simple identity $\frac{1}{u}=\frac{k}{u(u+k)}+\frac{1}{u+k}$; for $u=n(\alpha+r)+\beta+s+k$ to (2.5),we obtain,

$$ S=\sum_{n=0}^{\infty}\frac{k\,{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)}\times\frac{1}{\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

$$ +\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+2k)\}((n+j)!)} $$

$$ S=\sum_{n=0}^{\infty}\frac{k\{n(\alpha+r)+\beta+s\}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

$$ \times\frac{1}{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

$$ +\sum_{n=0}^{\infty}\frac{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} $$

$$ \times\frac{1}{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

using equation (1.9), we arrived at

$$ S=\sum_{n=0}^{\infty}\frac{k\{n(\alpha+r)+\beta+s\}_p(\gamma)_{(n+j)q,k}\,z^n}{\frac{k^3}{p^3}\,{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} $$

$$ +\sum_{n=0}^{\infty}\frac{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}_p(\gamma)_{(n+j)q,k}\,z^n}{\frac{k^3}{p^3}\,{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} $$

this reduces to

$$S\frac{k^3}{p^3} = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+p)+\beta+s+3k)((n+j)!)}$$

$$\times[n^2(\alpha+r)^2 + 2n(\alpha+r)(\beta+s+k) + (\beta+s)(\beta+s+2k)] \tag{2.6}$$

We now express each summation in the right hand side of (2.6) as follows:

$$\frac{d}{dz}[z\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)] = \sum_{n=0}^{\infty} \frac{(n+1)\,{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}$$

$$z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) = \sum_{n=0}^{\infty} \frac{(n+1)\,{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}$$

$$z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) = \sum_{n=0}^{\infty} \frac{n\,{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} \tag{2.7}$$

Again,

$$\frac{d^2}{dz^2}[z^2\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)] = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)\,{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} \tag{2.8}$$

On simplifications

$$\frac{d^2}{dz^2}[z^2\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)]$$

$$= z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + 4z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + 2\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) \tag{2.9}$$

from equation (2.8) and (2.9), immediately leads to

$$\sum_{n=0}^{\infty} \frac{\{n^2\}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)} = z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)$$

$$+ 4z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) - 3\sum_{n=0}^{\infty} \frac{\{n\}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}$$

using equation (2.7), follows

$$\sum_{n=0}^{\infty} \frac{\{n^2\}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)((n+j)!)}$$

$$= z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + z\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z), \tag{2.10}$$

applying equation (2.4), (2.7) and (2.10) to (2.6), we finally arrived at

$$\frac{k^3}{p^3}S = (\alpha+r)^2 z^2\,{}_p^j\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + \Big[(\alpha+r)^2 + (\alpha+r)(2\beta+2s+2k)\Big]z$$

$$\times\,{}_p^j\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + (\beta+s)(\beta+s+2k)\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)$$

**Theorem 2.2** For $k,p \in \mathbb{R}^+ \setminus \{0\}$; $\alpha+r, \beta+s+k, \gamma \in \mathbb{C}\setminus k\mathbb{Z}^-$; $R(\alpha+r) > 0, R(\beta+s+k) > 0, R(\gamma) > 0, q > 0, j \in \mathbb{N}_0$, we get

$$\int_0^1 t^{\beta+s+k-1}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s}(t^{\alpha+r})dt = \frac{p}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(1) - \frac{p^2}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(1) \tag{2.11}$$

**Proof:** Put $z = 1$ in equation (2.4) and (2.5), yields

$$S = \frac{p}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(1) - \frac{p^2}{k}\,{}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(1)$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k\}((n+j)!)} \tag{2.12}$$

consider the integral,

$$A \equiv \int_0^1 t^{\beta+s+k-1}\, {}_p^jE^{\gamma,q}_{k,\alpha+r,\beta+s}(t^{\alpha+r})dt$$

using the equation (1.12), gives

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)((n+j)!)} \int_0^1 t^{n(\alpha+r)+\beta+s+k-1}dt$$

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k\}((n+j)!)}$$

from equation (2.12), we have the desired result.

**Theorem 2.3** For $k,p \in \mathbb{R}^+ \setminus \{0\}; \alpha,\beta,\gamma,\delta \in \mathbb{C}; R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, R(\delta) > 0\, and\, q > 0, j \in \mathbb{N}_0$ then

$$p^{\delta}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta+\delta k}(z) = \frac{1}{\Gamma(\delta)} \int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\delta-1}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(z\, u^{\frac{\alpha}{k}})du \tag{2.13}$$

**Proof :** Consider the right side integral and using equation (1.12), we have

$$A \equiv \frac{1}{\Gamma(\delta)} \int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\delta-1}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(z\, u^{\frac{\alpha}{k}})du$$

$$A \equiv \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)} \int_0^1 u^{\frac{\alpha n+\beta}{k}-1}(1-u)^{\delta-1}du$$

using the definition of Beta function, yields

$$A \equiv \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)} \frac{\Gamma(\frac{\alpha n+\beta}{k})\Gamma(\delta)}{\Gamma(\frac{\alpha n+\beta}{k}+\delta)},$$

applying equation (1.5), one can write

$$A \equiv \sum_{n=0}^{\infty} \frac{p^{\delta}\, {}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta+\delta k)((n+j)!)} = p^{\delta}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta+\delta k}(z)$$

**Theorem 2.4** For $k,p \in \mathbb{R}^+ \setminus \{0\}; \beta,\gamma \in \mathbb{C}; R(\beta) > 0, R(\gamma) > 0\, and\, \alpha, q > 0, j \in \mathbb{N}_0$, then

$${}_p^jE^{\gamma,q}_{k,k\alpha,\beta}(z) = \frac{{}_p(\gamma)_{jq,k}}{{}_p\Gamma_k(\beta)} \prod_{i=1}^{q}\prod_{l=1}^{\alpha} \frac{\Gamma(b_l)}{\Gamma(a_i)\Gamma(b_l-a_i)} \int_0^1 u^{a_i-1}(1-u)^{b_l-a_i-1} E_{1,j+1}\Big(uz\frac{p^{q-\alpha}q^q}{\alpha^{\alpha}}\Big)du \tag{2.14}$$

Where $a_i = \dfrac{\frac{\gamma}{k}+qj+i-1}{q}$ and $b_l = \dfrac{\frac{\beta}{k}+l-1}{\alpha}$

**Proof :** Using definition of j-generalized p-k Mittag- Leffler function, from equation (1.12),

$$A \equiv {}_p^jE^{\gamma,q}_{k,k\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, z^n}{{}_p\Gamma_k(nk\alpha+\beta)((n+j)!)}$$

using relation (1.11),we have

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{jq,k}\, {}_p(\gamma+jqk)_{nq,k}\, z^n}{{}_p(\beta)_{n\alpha,k}\, {}_p\Gamma_k(\beta)((n+j)!)} = \sum_{n=0}^{\infty} D \frac{{}_p(\gamma)_{jq,k}\, z^n}{{}_p\Gamma_k(\beta)((n+j)!)} \tag{2.15}$$

Where $D \equiv \dfrac{{}_p(\gamma+jqk)_{nq,k}}{{}_p(\beta)_{n\alpha,k}}$
using equation (1.6), we have

$$D \equiv \frac{{}_p(\gamma+jqk)_{nq,k}}{{}_p(\beta)_{n\alpha,k}} = \frac{p^{n(q-\alpha)}(\frac{\gamma+jqk}{k})_{qn}}{(\frac{\beta}{k})_{n\alpha}}$$

using the relation given by equation (1.7), we get

$$D \equiv \frac{p^{(q-\alpha)n}q^{qn}\prod_{i=1}^{q}(\frac{\frac{\gamma}{k}+jq+i-1}{q})_n}{\alpha^{\alpha n}\prod_{l=1}^{\alpha}(\frac{\frac{\beta}{k}+l-1}{\alpha})_n}$$

$$let\ a_i = \frac{\frac{\gamma}{k}+jq+i-1}{q}\ and\ b_l = \frac{\frac{\beta}{k}+l-1}{\alpha}$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{(a_i)_n}{(b_l)_n}$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(a_i+n)\Gamma(b_l)}{\Gamma(b_l+n)\Gamma(a_i)}$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(b_l)}{\Gamma(b_l-a_i)\Gamma(a_i)}\frac{\Gamma(a_i+n)\Gamma(b_l-a_i)}{\Gamma(b_l-a_i+a_i+n)}$$

using the definition of Beta function, immediately gives

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(b_l)}{\Gamma(b_l-a_i)\Gamma(a_i)}\int_0^1 u^{a_i+n-1}(1-u)^{b_l-a_i-1}du \tag{2.16}$$

from equation (2.15) and (2.16), we arrived

$$A \equiv \frac{{}_p(\gamma)_{jq,k}}{{}_p\Gamma_k(\beta)}\prod_{i=1}^{q}\prod_{l=1}^{\alpha}\frac{\Gamma(b_l)}{\Gamma(b_j-a_i)\Gamma(a_i)}\int_0^1 u^{a_i-1}(1-u)^{b_j-a_i-1}\sum_{n=0}^{\infty}\frac{(uz)^n}{(n+j)!}(\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n du$$

**Theorem 2.5** For $k,p \in \mathbb{R}^+ \setminus \{0\}; \alpha,\beta,\gamma \in \mathbb{C}; R(\alpha)>0, R(\beta)>0, R(\gamma)>0$ and $q>0, j\in\mathbb{N}_0$ then

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{p^{jq}}{\Gamma(\frac{\gamma}{k})}\int_0^\infty e^{-t}\,t^{(\frac{\gamma}{k}+jq-1)}\,{}_p^jE_{k,\alpha,\beta}^{1,0}(zt^qp^q)dt \tag{2.17}$$

**Proof:** Using definition of j-generalized p-k Mittag-Leffler function, equation (1.12), written as

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}\,z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)}$$

using equation (1.5) and (1.6), leads to

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty}\frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)}\frac{p^{(n+j)q}\Gamma(\frac{\gamma}{k}+q(n+j))}{\Gamma(\frac{\gamma}{k})}$$

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty}\frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)}\frac{p^{q(n+j)}}{\Gamma(\frac{\gamma}{k})}\int_0^\infty e^{-t}\,t^{(\frac{\gamma}{k}+q(n+j)-1)}dt$$

i.e.

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{p^{jq}}{\Gamma(\frac{\gamma}{k})}\int_0^\infty e^{-t}\,t^{(\frac{\gamma}{k}+jq-1)}\sum_{n=0}^{\infty}\frac{z^n\,p^{qn}\,t^{qn}}{{}_p\Gamma_k(n\alpha+\beta)((n+j)!)}dt$$

and hence,

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{p^{jq}}{\Gamma(\frac{\gamma}{k})}\int_0^\infty e^{-t}\,t^{(\frac{\gamma}{k}+jq-1)}\,{}_p^jE_{k,\alpha,\beta}^{1,0}(zt^qp^q)dt.$$

# 3 Integral Transform of the j-generalized P-K Mittag-Leffler Function

In this section, Mellin-Barnes integral representation of j-generalized p-k Mittag-Leffler function, relationship with Fox H-function and Wright hypergeometric function have been discussed. The well known integral transforms viz Euler Beta Transform, Laplace Transform and Mellin Transform of j-generalized p-k Mittag-Leffler function also discussed.

**Mellin-Barnes integral representation of the j-generalized p-k Mittag-Leffler function.**

**Theorem 3.1** Let $k, p \in \mathbb{R}^+ \setminus \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and\, q > 0, j \in \mathbb{N}_0$, then the j-generalized p-k Mittag-Leffler function is represented by the Mellin-Barnes integral as,

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)} (-zp^{q-\frac{\alpha}{k}})^{-s} ds \tag{3.1}$$

Where $| \, argz \, | < \pi$; the contour integration beginning at $-i\infty$ and ending at $+i\infty$, and indented to separate the poles of the integrand as $s = -n$ for every $n \in N_0$(to the left) from those at $s = \frac{\frac{\gamma}{k}+n}{q}$ for every $n \in N_0$ (to the right).

**Proof** Using theorem of residues, the right hand side of equation (3.1) written as

$$I \equiv \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)} (-zp^{q-\frac{\alpha}{k}})^{-s} ds$$

$= 2\pi i$[sum of the residues at the poles $s = 0, -1, -2, ...$]

Above equation, reduces to

$$I \equiv \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \underset{s=-n}{Re} \quad (s+n)\Big[\frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)}\Big](-zp^{q-\frac{\alpha}{k}})^{-s}$$

This can be written as

$$I \equiv \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \lim_{s\to -n} \Big[\frac{\pi(s+n)}{sin\pi s} \frac{\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(1+j-s)\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})}\Big](-zp^{q-\frac{\alpha}{k}})^{-s}$$

Above equation, leads to

$$I \equiv \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \Big[\frac{\Gamma(\frac{\gamma}{k}+jq+qn)}{\Gamma(1+j+s)\Gamma(\frac{\beta}{k}+\frac{\alpha n}{k})}\Big](zp^{q-\frac{\alpha}{k}})^n$$

using equations (1.5) and (1.6), we have,

$$I \equiv {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z)$$

**Relationship with Fox H-function**

**Theorem 3.2** Let $k, p \in \mathbb{R}^+ \setminus \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and\, q > 0, j \in \mathbb{N}_0$ then the j-generalized p-k Mittag-Leffler function is given in the form of Fox H-function as,

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} H_{2,3}^{1,2}\left[-zp^{q-\frac{\alpha}{k}} \Bigg| \begin{matrix} (0,1),(1-\frac{\gamma}{k}-jq,q); \\ \\ (0,1),(1-\frac{\beta}{k},\frac{\alpha}{k}),(-j,1); \end{matrix}\right] \tag{3.2}$$

**Proof.** Using the equations (3.1) and (1.14), we get the desired result.

**Relationship with Wright hypergeometric function**

**Theorem 3.3** Let $k, p \in \mathbb{R}^+ \setminus \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and\, q > 0, j \in \mathbb{N}_0$, then the j-generalized p-k Mittag-Leffler function is given in the form of Wright hypergeometric function, as.

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{jq-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})}\, {}_2\Psi_2 \left[ \begin{array}{c} (1,1)\ (\frac{\gamma}{k}+jq, q); \\ \hfill zp^{q-\frac{\alpha}{k}} \\ (\frac{\beta}{k}, \frac{\alpha}{k})(1+j,1); \end{array} \right] \tag{3.3} $$

**Proof.** Using the equations (3.1) and (1.13), we get the desired result.

**Euler(Beta) Transform, Laplace Transform and Mellin Transform of j-generalized p-k Mittag-Leffler function**

**Theorem 3.4** Let $k, p \in \mathbb{R}^+ \setminus \{0\}$; $a, b, \sigma \in \mathbb{C}$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$ and $q > 0, j \in \mathbb{N}_0$, then Euler Beta Transform of j-generalized p-k Mittag-Leffler function, is given by,

$$ \int_0^1 z^{a-1}(1-z)^{b-1}\, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)dz = \frac{kp^{jq-\frac{\beta}{k}}\Gamma(b)}{\Gamma(\frac{\gamma}{k})}\, {}_3\psi_3 \left[ \begin{array}{c} (1,1), (\frac{\gamma}{k}+qj, q), (a,\sigma); \\ \hfill xp^{q-\frac{\alpha}{k}} \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (1+j,1), (a+b,\sigma); \end{array} \right] \tag{3.4} $$

**Proof** Consider, the left side integral and using equation (3.4), we have

$$ I \equiv \int_0^1 z^{a-1}(1-z)^{b-1}\, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)dz $$

$$ \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, x^n}{{}_p\Gamma_k(n\alpha+\beta)\,(n+j)!} \int_0^1 z^{\sigma n+a-1}(1-z)^{b-1}dz, $$

using definition of Beta function, we obtain

$$ A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}\, x^n}{{}_p\Gamma_k(n\alpha+\beta)\,(n+j)!} B(\sigma n+a, b) $$

using equation (1.5),(1.6) and (1.13), gives (3.4)

**Theorem 3.5** The Laplace transform of j-generalizedp-k Mittag-Leffler function, is given by,

$$ \int_0^\infty z^{a-1}e^{-zs}\, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)dz = \frac{kp^{jq-\frac{\beta}{k}}\, s^{-a}}{\Gamma(\frac{\gamma}{k})}\, {}_3\psi_2 \left[ \begin{array}{c} (1,1), (\frac{\gamma}{k}+qj, q), (a,\sigma); \\ \hfill \frac{xp^{q-\frac{\alpha}{k}}}{s^\sigma} \\ (\frac{\beta}{k}, \frac{\alpha}{k})(1+j,1); \end{array} \right] \tag{3.5} $$

Where $k, p \in \mathbb{R}^+ \setminus \{0\}$; $a, \sigma \in \mathbb{C}$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$ and $q > 0, j \in \mathbb{N}_0$, and $\mid \frac{x}{s^\sigma} \mid < 1$.

**Proof** Consider the right side integral and using equation(1.12), we have

$$ I \equiv \int_0^\infty z^{a-1}e^{-zs}\, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)dz $$

$$ \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{x^n}{(n+j)!} \int_0^\infty z^{n\sigma+a-1}\, e^{-zs}dz $$

using definition of gamma function, yields

$$ A \equiv s^{-a} \quad \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)\,(n+j)!} \Gamma(\sigma n+a)(\frac{x}{s^\sigma})^n $$

using equation (1.5),(1.6) and (1.13), arrived at (3.5)

**Theorem 3.6** The Mellin transform of j-generalized p-k Mittag-Leffler function, is given by,

$$\int_0^\infty t^{s-1}\,{}_p^jE_{k,\alpha,\beta}^{\gamma,q}(-wt)dt = \frac{kp^{qj-\frac{\beta}{k}}\Gamma(s)\,\Gamma(1-s)\,\Gamma(\frac{\gamma}{k}+qj-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(\frac{\gamma}{k})\Gamma(1+j-s)}(\frac{p^{\frac{\alpha}{k}-q}}{w})^s \tag{3.6}$$

Where $k,p \in \mathbb{R}^+ \setminus \{0\}$; $a,\sigma \in \mathbb{C}$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(s) > 0$ and $q > 0, j \in \mathbb{N}_0$.

**Proof** Putting z = -wt in equation (3.1), we have

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(-wt) = \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})}\int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)}(-wtp^{q-\frac{\alpha}{k}})^{-s}ds$$

this can be written as

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(-wt) = \frac{kp^{jq-\frac{\beta}{k}}}{2\pi i\Gamma(\frac{\gamma}{k})}\int_L f^*(s)(t)^{-s}ds. \tag{3.7}$$

where,

$$f^*(s) = \frac{\Gamma(s)\Gamma(1-s)\,\Gamma(\frac{\gamma}{k}+jq-qs)}{\Gamma(\frac{\gamma}{k})\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(1+j-s)}(-wp^{q-\frac{\alpha}{k}})^{-s}$$

using equation (1.17),(1.18) and (3.7), immediately leads to (3.9).

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