Graph Analysis: Edge Fixing and Its Impact on Detour Number Metrics

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ABSTRACT

We introduce the concept of the total edge fixing edge-to-vertex detour setof a connected graph G. Let e be an edge of a graph G. A set $S(e) \subseteq E(G) - \{e\}$is called an edge fixing edge-to-vertex detour set of a connected graph G if every edge of G lies on an $e - f$ detour, where $f \in S(e)$. The edge fixing edge-to-vertex detour number defev(G) of G is the minimum cardinality of its edge fixing edge-to-vertex detour sets and any edge fixing edge-to-vertex detour set of cardinality $dn_{efev}(G)$ is an d_{efev}-set of G. Connected graphs of order p with edge fixing edge-to-vertex detour number 1 or $q - 1$ are characterized. Theedge fixing edge-to-vertex detour number for some standard graphs are determined. It is shown that for every pair of positive integers with $2 \leq a \leq b$, there exists a connected graph G such that $dn_{ev}(G) = a$ and $dn_{efev}(G) = b$, for some edge $e \in E(G)$..

Keywords: *detour set ,edge-to-vertex detour set , edge fixing edge –to-vertex detour set,edge fixing edge - to vertex detour set,edge fixing edge - to - vertex detour number.*

Mathematical subject classification 05C12.

I. INTRODUCTION

For a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1,4].For vertices u and v in a connected graph G, the *detour distance*$D(u, v)$ is the length of the longest $u - v$ path in G.A $u - v$ path of length $D(u, v)$ is called a $u - v$*detour*. It is known that the detour distance is a metric on the vertex set $V(G)$. The *detour eccentricity*$e_D(v)$ of a vertex v in G is the maximum detour distance form v to a vertex of G.The detour *radius*,$rad_D G of G$is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$of G is the maximum detour eccentricity among the vertices of G. These concept were studied by Chartrand et al.[2]. Let $G = (V, E)$ be a connected graph with at least 3 vertices. A set $S \subseteq E$ is called an *edge-to-vertex detour set* if every vertex of G is either incident with an edge of S or lies on a detour joining a pair of edges of S. The *edge-to-vertex detour number*$d_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex detour sets and any edge-to-vertex detour set of cardinality $d_{ev}(G)$ is an *edge-to-vertex detour*d_{ev}*-set* of G.

Theorem 1.1[6]
Every pendant edge of a connected graph G belongs to every edge-to-vertex detour set of G.

Theorem 1.2[6]
For any non-trivial tree T with pendant edges, $d_{ev}(T) = k$ and the set of all pendant edges of T is the unique minimum edge-to- vertex detour set of T.

II. THE EDGE FIXING EDGE-TO-VERTEX DETOUR

Number of a Graph
Definition 2.1
Let e be an edge of a graph G. A set $S(e) \subseteq E(G) - \{e\}$ is called an *edge fixing edge-to-vertex detour set* of a connected graph G if every edge of G lies on an $e - f$ detour, where $f \in S(e)$. The *edge fixing edge-to-vertex detour number* $d_{efev}(G)$ of G is the minimum cardinality of its edge fixing edge-to-vertex detour sets and any edge fixing edge-to-vertex detour set of cardinality $d_{efev}(G)$ is an d_{efev} -set of G.
Example 2.2
For the graph$G$ given in Figure 2.1, the edge fixing edge-to-vertex detour sets of each edge of G is given in the following Table 2.1.

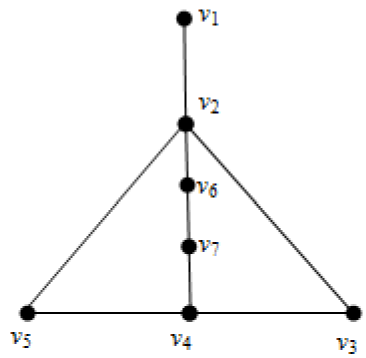

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Figure 2.1

Table 2.1

Fixing Edge (e)	**Minimum edge fixing edge-to-vertex detour sets ($S(e)$)**	$d_{efev}(S(e))$
v_1v_2	$\{v_2v_6\}$,$\{v_6v_7\}$	1
v_2v_3	$\{v_1v_2, v_6v_7\}$	2
v_3v_4	$\{v_1v_2, v_4v_5\}$	2
v_4v_5	$\{v_1v_2, v_3v_4\}$	2
v_2v_5	$\{v_1v_2, v_6v_7\}$	2
v_6v_2	$\{v_1v_2\}$	1

Remark 2.3
For a connected graph G, the edge e of G does not belong to the edge fixing edge-to- vertex detour set $S(e)$. Also the edge fixing edge-to- vertex detour set of an edge e is not unique. For the graph G given in Figure 6.1, the edge fixing edge-to- vertex detour sets of the edge $v_1v_2$are $\{v_6v_7\}$, $\{v_2v_6\}$.

III. SOME RESULTS ON THE EDGE FIXING EDGE-TO-VERTEXDETOURNUMBER OF A GRAPH

Theorem 2.4
Let e be an edge of G. Let f be a pendant edge of a connected graph Gsuch that $e \neq f$. Then every edge fixing edge-to- vertex detour set of $e$of G contains f.

Proof. Since $e \neq f, f$ is a terminal edge of a detourhence $f$belongs to every edge fixing edge-to- vertex detour set of $e$of$G$. ■

Theorem 2.5
Let G be a connected graph and $S(e)$ be an edge fixing edge-to- vertex detour set of $e$of G. Let f be a non-pendant cut edge of $G$and letG_1 and G_2 be the two component of $G - \{f\}$. If $e = f$, then each of the two component of $G - \{f\}$ contains an element of $S(e)$. If $e \neq f$, then $S(e)$ contains at least one edge of component of $G - \{f\}$ where e does not lie.

Proof. Let $f = uv$. Let $G_1$and G_2 be the two component of $G - \{f\}$ suchthat$u \in V(G_1)$and $\in V(G_2)$.Let $e = f$. Suppose that $S(e)$ does not contain any element of G_1. Then$S(e) \subseteq E(G_2)$.Let h be an edge of $E(G_1)$. Then h must lie on an e– f'detourforsome $f' \in S(e)$. But such a detour P: $v, v_1, v_2,\ldots,v_l, v, u, u_1, u_2,\ldots, u_s, u, v, v_1, v_2,\ldots, v'$ where

$v_1, v_2, \ldots, v_l \in V(G_2)$, $u_1, u_2, \ldots, u_s \in V(G_1)$ and v' is an end of f' has the cut-edge f twice, hence it is a contradiction. This proves the theorem.By similar argument, we can prove that if $e \neq f$, then $S(e)$ contains at least one edge from a component of $G - \{f\}$ where e does not lie. ■

Theorem 2.6
Let G be a connected graph and $S(e)$ be a minimum edge fixing edge-to- vertex detour set of an edge $e$of G. Then no non-pendant cut-edge of G belongs to $S(e)$.

Proof. Let $S(e)$ be an edge fixing edge-to- vertex detour set of an edge $e = uv$of G. Let $f = u'v'$be a non-pendant cut-edge of G such that $f \in S(e)$. Since $e \neq f$, let G_1 and G_2 be the two component of G - $\{f\}$ such that $u' \in V(G_1)$ and $v' \in V(G_2)$. By Theorem6.5, G_1 contains an edge $xy$and G_2 contains an edge $x'y'$ where $xy, x'y' \in S(e)$. Let $S'(e) = S(e)$- $\{f\}$. We claim that $S'(e)$ is an edge fixing edge-to- vertex detour set of an edge $e$of G.

Case 1. Suppose that $e = xy$ is an edge in G_1 and $x'y'$is an edge in G_2. Let h be avertexof G. Assume without loss of generality that $h = wz$belongs to G_1. Since $u'v'$is a cut-edge of G, every path joining an edge of G_1 with an edge of G_2 contains the edge $u'v'$. Suppose that h is adjacent with $u'v'$or the edge xy of $S(e)$ or that lies on a detour joining $xy$and $u'v'$. If h is adjacent with $u'v'$,then $z = u'$. Let $P : x, y, y_1, y_2, \ldots, w, z = u'$beaxy $-u'v'$ detour. Let $Q: u', v', v_1', v_2', \ldots, x', y'$ a $u'v'$-$x'y'$detour. Then, it is clear that P followed by $u'v'$and Q is a xy-$x'y'$detour. Thus h lies on the xy-$x'y'$ detour. If $h$is adjacent with xy, then there is nothing to prove. If h lies on a xy-$x'y'$detour, say $x, y, v_1, v_2, \ldots, w, z, \ldots, u', v'$, then let $u', v', v_1', v_2', \ldots, y'$ be $u'v'$-$x'y'$detour. Thenclearly$x, y, v_1, v_2, \ldots, w, z, \ldots,$ $u', v', v_1', v_2', \ldots, x', y'$is a xy-$x'y'$detour. Thus h lies on a detour joining$xy$and an element of $S'(e)$. Thus we have proved that an edge that is adjacent with $u'v'$or an edge of $S(e)$ or that lies on a detour joining xy and $u'v'$of $S(e)$ also is adjacent with an edge of $S'(e)$ or lies on a detour joining e and an edge of $S'(e)$. Hence it follows that $S'(e)$ is an edge fixing edge-to- vertex detour set of an edge e of G such that $|S'(e)| = |S(e)|$ - 1, which is a contradiction to the minimality of $S(e)$.

Case 2. Suppose that $e = xy \in G_2$.The proof is similar to that of Case 1. Hence the theorem follows. ■

Theorem 2.7
For any non-trivial tree T with $k$end edges,

$$d_{efev}(G) = \begin{cases} k-1 & \text{if } e \text{ is an end edge of } G \\ k & \text{if } e \text{ is an internal edge of } G \end{cases}.$$

Proof. This follows from Theorem 2.4 and Theorem 2.6. ■

Theorem 2.8
For the graph $G = C_p (p \geq 4), d_{efev}(G) = 1$, for any edge e of $E(G)$.

Proof. Let $C_p: v_1, v_2, v_3, \ldots, v_p$ be the cycle. Let $e$be an edge of $C_p$and f be an edge adjacent to e. Then it follows that $\{f\}$ is an edge fixing edge-to- vertex detour set of an edge $e$of C_p. Hence $d_{efev}(C_p) = 1$. ■

Theorem 2.9
For the complete graph $K_p (p \geq 4), d_{efev}(G) = 1$ for every edge in $E(G)$.
Proof. We observe that all the edges of $K_p$can be considered as the edges of $C_p$and every edge joining the points of C_p. Let $e$be an edge of $C_p$and f be an edge adjacent to e. Then it follows that $\{f\}$ is an edge fixing edge-to-vertex detour set of an edge e of C_p.Hence $d_{efev}(K_p) = 1$.■

Theorem 2.10
Let G be a connected graph with at least three vertices. Then $1 \leq d_{efev}(G) \leq q - 1$.

Proof. For any edge e in G, an edge fixing edge-to-vertex detour set needs at least one edge of G so that $d_{efev}(G) \geq 1$. For an edge $e \in E(G)$, $E(G) - \{e\}$ is an edge fixing edge-to-vertex detour set of e of G so that $d_{efev}(G) \leq q - 1$. Therefore $1 \leq d_{efev}(G) \leq q - 1$. ■

Remark 2.11
The bounds in Theorem 2.10 are sharp. For the cycle $G = C_p$ $(p \geq 4)$, for an edge e, any edge which is adjacent to$e$ is its minimum edge fixing edge-to-vertex detour set of e of G so that $d_{efev}(G) = 1$. For the star $G = K_{1,q}$, for an edge e, the set of edges $E(G) - \{e\}$ is the unique edge fixing edge-to-vertex detour set of e of G so that $d_{efev}(G) = q - 1$. Thus the star $K_{1,q}$ has the largest possible edge fixing edge-to-vertex detour number $q - 1$ and the cycle $G = C_p$ $(p \geq 4)$, has the smallest edge fixing edge-to-vertex detour number 1. Also the bounds in Theorem 2.10 is strict. For the graph G given in Figure 2.1, for the edge $e = v_3 v_4, d_{efev}(G) = 2$ so that $1 < d_{efev}(G) < q - 1$.

Theorem 2.12
Let G be a connected graph of size $q \geq 3$, such that G is neither a star nor a double star. Then $d_{efev}(G) \leq q - 2$ for every $e \in E(G)$.

Proof.
Case 1. Suppose that G is a tree such that $G$is neither a star nor a double star. Then by Theorem 2.7,$d_{efev}(G) \leq q - 2$, for every $e \in E(G)$.

Case 2. Suppose that G is not a tree. Then G contains at least one cycle, say C. Let e be an edge of G

Subcase 2a. Suppose that$e \in E(C)$. Then $S(e) = E(G) - E(C)$is an edge fixing edge-to-vertex detour set of an edge e of G so that $d_{efev}(G) \leq q - 2$.

Subcase 2b. Suppose that$e \notin E(C)$. Then setting $S(e) = E(G) - E(C) - \{e\}$ and by the similar argument in Subcase2a we can prove that $d_{efev}(G) \leq q - 2$. Hence the proof. ■

Remark 2.13
The bound in Theorem2.12 is sharp. For the graph $G = C_3$, it is easily verified that $d_{efev}(G) = q - 2$ for every edge $e$of G.

Theorem 2.14
Let Gbe a connected graph of size $q \geq 2$ and $e \in E(G)$. Then $d_{efev}(G) = q - 1$ if and only if e is an edge of $K_{1,q}$ or e is an internal edge of a double star.

Proof. Let G be a connected graph. If e is an edge of $K_{1,q}$, then by Theorem 2.7,$d_{efev}(G) = q - 1$. If e is an internal edge of a double star, then by Theorem 2.7,$d_{efev}(G) = q - 1$.
Conversely, let $d_{efev}(G) = q - 1$ for an edge $e \in E(G)$. Suppose that$e$is neither an edge of $K_{1,q}$nor an internal edge of a double star. Then by Theorem 2.12,$d_{efev}(G) = q - 2$, which is a contradiction.Therefore$e$ is an edge of $K_{1,q}$ or $e$is an internal edge of a double star.■

Theorem 2.15
Let G be a connected graph with $q \geq 4$, which is not a cycle and not a tree and let $C(G)$ be the length of the longest cycle. Then $d_{efev}(G) \leq q - C(G) + 1$ for some $e \in E(G)$.

Proof. Let $C(G)$ denote the length of the longest cycle in G and C be the cycle of length k.
Let $C: v_1, v_2, v_3, \ldots, v_k$ be a cycle, $k \geq 3$. Since G is not a cycle, there exists a vertex $v$in G such that v is not a vertex of C and which is adjacent to v_1, say. Let e be an edge of C. Let $S(e) = E(G) - \{E(C) - e\}$. Clearly $S(e)$ is an edge fixing edge-to-vertex detour set of $e$of$G$ so that $d_{efev}(G) \leq q - C(G) + 1$. ■

Theorem 2.16
Let G be a connected graph of size $q \geq 3$ which is not a double star and $d_{efev}(G) = q - 2$for some edge $e$of G. Then G is unicyclic.

Proof. Suppose that G is not unicyclic. Then G contains more than one cycle.

Let C_1 and C_2 be the two cycles of G. By Theorem 2.15, $|C_1| = |C_2| = 3$.

Case 1. Suppose that C_1 and C_2have exactly one vertex, say, $v$in common.
Let $e = uv$ be an edge of $C_1$and let $S(e) = E(G) - E(C) - \{e, f\}$, where $f = vw$, where $w \in V(C_2)$. Then $S(e)$ is an edge fixing edge-to-vertex detour set of an edge e of G so that $d_{efev}(G) = q - 3$, which is a contradiction.

Case 2. Suppose that C_1 and C_2have a common edge, say, uv.
Let $e = uv$ and let $S(e) = E(G) - \{e, uw, uz\}$, where $w \in V(C_1)$ and $z \in V(C_2)$. Then $S(e)$ is an edge fixing edge-to-vertex detour set of $e$of$G$ so that $d_{efev}(G) = q - 3$, which is a contradiction.

Case 3. Suppose that C_1 and C_2 are connected by a path P.
Suppose that $e = xu$ be an edge of C_1, where x is a vertex common to C_1 are P and let $S(e) = E(G) - \{e, xu_1, xx_1, f\}$,where $xu_1 \in E(C_1)$ such that $u \neq u_1, xx_1 \in E(P)$ and $f \in E(C_2)$. Then clearly $S(e)$ is an edge fixing edge-to-vertex detour set of e of G so that $d_{efev}(G) \leq q - 4$, which is a contradiction. ■

Theorem 2.17
For a connected graph G, $d_{ev}(G) \leq d_{efev}(G) + 1$.

Proof. Let e be an edge of G and $S(e)$ be the minimum edge fixing edge-to-vertex detour set of $e$of G. Then $S(e) \cup \{e\}$ is an edge-to-vertex detour set of e of G so that $d_{ev}(G) \leq |S(e) \cup \{e\}| = d_{efev}(G) + 1$. ■

Remark 2. 18
The bound in Theorem 2.17 is sharp. For the cycle C_p, $d_{efev}(C_p) = 1$for every $e \in E(G)$ and $d_{ev}(G) = 2$ so that $d_{ev}(G) = d_{efev}(G) + 1$.Also the inequality in the Theorem 2.17 strict. For the graph Ggiven in Figure 2.2, let $e = u_3u_4$. Then $S(e) = \{u_1u_2, u_7, u_8\}$ is an edge fixing edge-to-vertex detour set of e of G so that $d_{efev}(G) = 2$. Also $d_{ev}(G) = 2$. Hence $d_{ev}(G) < d_{efev}(G) + 1$.

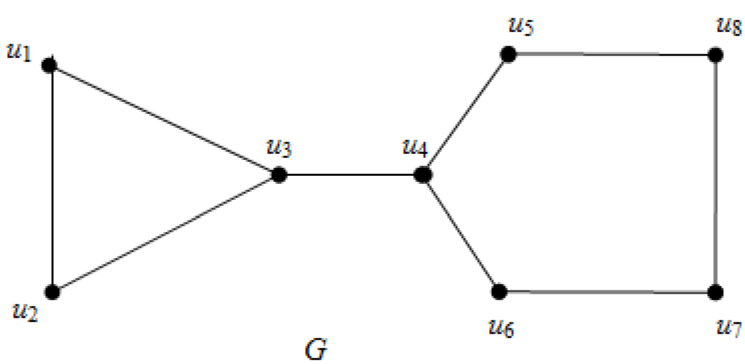

Figure 2.2

Theorem 2.19
For positive integers $R, D$and $l \geq 2$with $R < D \leq 2R$, there exists a connected graph G with $rad(G) = R, diam(G) = D$ and $d_{efev}(G) = l$ for some $e \in E(G)$.

Proof. When $R = 1$, we let $G = K_{1,l}$. Then the result follows from Theorem 2.7. Let $R \geq 2$. Let $C_{R+1}: v_1, v_2, \dots, v_{R+1}$ be a cycle of length $R + 1$ and let $P_{D-R}: u_0, u_1, u_2, \dots, u_{D-R}$ be a path of length $D - R$. Let H be a graph obtained from C_{R+1} and P_{D-R} by identifying v_1 in C_{R+1} and $u_0$in P_{D-R}. Now add $l - 2$ new vertices $w_1, w_2, \dots, w_{l-2}$ to H and join each w_i $(1 \leq i < l - 2)$ to the vertex u_{D-R_1} and obtain the graph G as shown in Figure 2.3. Then $rad_D(G) = R$and $diam_D(G) = D$. Let $S = \{u_{D-R-1}u_{D-R}, u_{D-R-1}w_1, u_{D-R-1}w_2, \dots, u_{D-R-1}w_{l-2}\}$ be the set of end-edges of G. Let $e$be a non-pendant cut edge of G.By Theorem 2.4, S is a subset of every edge fixing edge-to-vertex detour set of G. It is clear that S is not an edge fixing edge-to-vertex detour set of G and so $d_{efev}(G) \geq l$. However $S \cup \{v_1v_2\}$ is an edge fixing edge-to-vertex detour set of e of G and so that $d_{efev}(G) = l$. ■

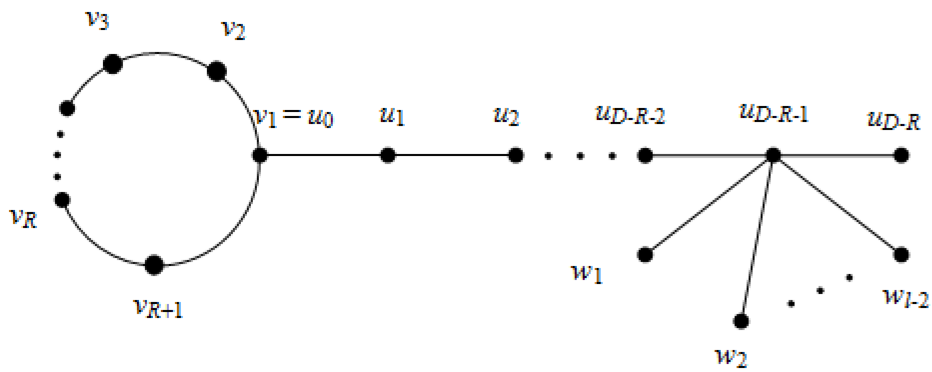

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Figure 2.3

Theorem 2.20
For any positive integer $a, 1 \leq a \leq q - 1$, there exists a connected graph G of size q such that $d_{efev}(G) = a$, for some edge $e \in E(G)$.

Proof. Let G be a connected graph.

Case 1. Let $a = q - 1$.
For the star $G = K_{1,q}$, by Theorem 6.7,$d_{efev}(G) = q - 1 = a$ for every edge $e \in E(G)$.

Case 2.$a = 1$
Let G be a path of length q and e be an pendant-edge of G. Then by Theorem 2.7, $d_{efev}(G) = 1 = a$.

Case 3.$1 < a < q - 1$
Let G be a tree with a end-edges and $q - a$ internal edges and let e be an internal edge of G. Then by Theorem 2.7, $d_{efev}(G) = a$. ∎
In view of Theorem 2.17, we have the following realization result.

Theorem 2.21
For every pair of positive integers with $2 \leq a \leq b$, there exists a connected graph G such that $d_{ev}(G) = a$ and $d_{efev}(G) = b$ for some edge $e \in E(G)$.

Proof. Let G be a connected graph.

Case 1.$a = b$
Let G be a double star with a end-edges and let e be the cut-edge of G. Then by Theorem 2.8, $d_{efev}(G) = a$. Also by Theorem 1.2, $d_{ev}(G) = a$.

Case 2.$2 \leq a < b$
Let $P : u_1, u_2, u_3, u_4, u_5, u_6, u_7$,be a path of order 7. Let $P_i : x_i y_i (1 \leq i \leq b - a + 1)$ be a copy of a path of order 2. Let H be a graph obtained from the path on PandP_i by joining u_1 with each $x_i (1 \leq i \leq b - a + 1)$ and $u_7$with $y_i (1 \leq i \leq b - a + 1)$. Let G be the graph obtained from H by adding new vertices $z_1, z_2, \dots, z_{a-1}$ and joining each $z_i (1 \leq i \leq a - 1)$with u_7.. The graph G is shown in Figure 2.4. First show that $d_{ev}(G) = a$. Let $S = \{z_1 u_7, z_2 u_7, \dots, z_{a-1} u_7\}$ be the set of all pendant-edges of G. By Theorem 1.1, S is a subset of every edge-t0-vertexdetour set of $e$of$G$. It is clear that S is not an edge-to-vertex detour set of G and so $d_{ev}(G) \geq a - 1$.However $S' = S \cup \{u_6 u_7\}$ is an edge-to-vertex detour set of G. Thus $d_{ev}(G) = a$. Let $e = u_1 x_1$. By Theorem2.4, $S = \{z_1 u_7, z_2 u_7, \dots, z_{a-1} u_7\}$ is a subset of every edge fixing edge-to-vertex detour set of e of G. It is clear that $S$is not an edge fixing edge-to-vertex detour set of $e$of G. It is easily verified that every edge fixing edge-to-vertex detour set

of e of Gcontains $x_i y_i$ $(2 \le i \le b - a + 1)$ and so $d_{efev}(G) \ge a - 1 + b - a + 1 = b$. Let $S(e) = S \cup \{x_1 y_1, x_2 y_2, \ldots, x_{b-a+1}, y_{b-a+1}\}$. Then$S(e)$ is an edge fixing edge-to-vertex detour set of e of G so that $d_{efev}(G)= b$. Hence the proof. ■

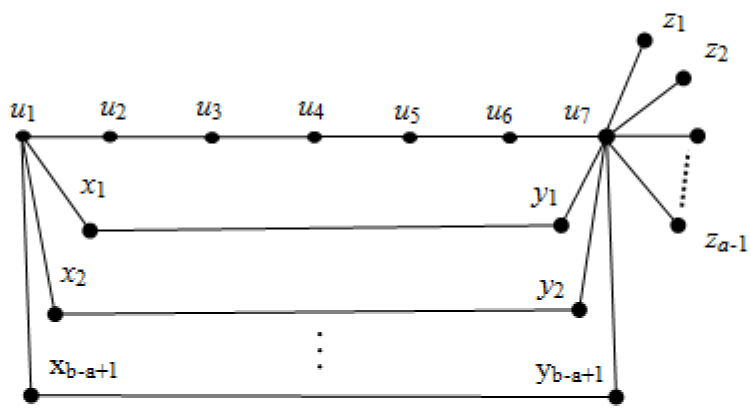

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Figure 2.4

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