On false belief in mathematics - or when a wrong solution does not mean a lack of knowledge

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Abstract

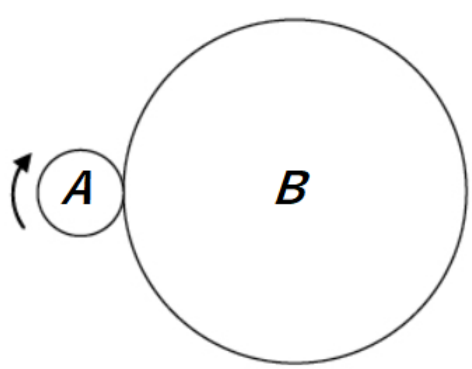
This paper explores the famous coin rotation paradox from the perspective of intuition and reasoning. Although solving the problem does not require advanced mathematical tools, it turns out that almost everyone gets the answer wrong. However, after learning the correct solution, everyone understands their mistake and it turns out that the mistake was not caused by a lack of proper substantive preparation.

We begin by explaining the nature of the problem and discussing the reasoning leading to both- the incorrect and correct answers. We then consider possible causes of incorrect solutions, from the perspective of mathematics teaching.

Keywords: coin rotation paradox, cycloid, epicycloid, hypocycloid, roulettes, rotation, misconception, false beliefs

Introduction

In the 1980s, the following question appeared on the S.A.T. exam (standardized test widely used for college admissions in the United States):



In the figure above, the radius of circle A is $\frac{1}{3}$ the radius of circle B. Starting from the position shown in the figure, circle A rolls around circle B. At the end of how many revolutions of circle A will the center of circle first reach its starting point?

(a) $\frac{3}{2}$ (b) 3 (c) 6 (d) $\frac{9}{2}$ (e) 9

Most candidates chose answer 3, simply dividing the circumference of the larger circle by the circumference of the smaller one. And it as was indicated as a correct answer on the answer key. Many may be surprised to learn that in fact is not. This problem became famous worldwide. On May 25, 1982, an article on the topic even appeared in the New York Times (available in the bibliography).

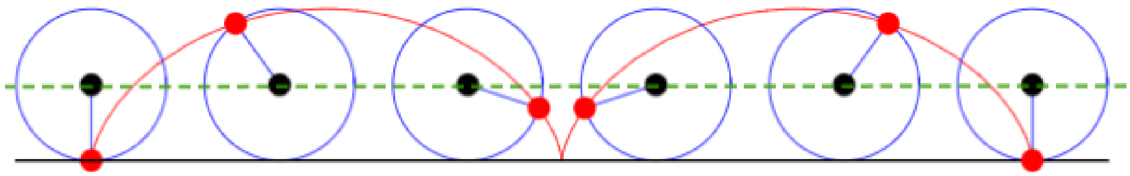
Due to the lack of a correct answer in the examination paper and the fact that the correct solution does not agree with the first intuition, this problem has been named coin rotation paradox. If we assume that circles A and B are congruent and have equal perimeters, intuition suggests that circle A should complete one full rotation to return to its starting point. However, a simple experiment shows that this is not the case (see the figure below).

Methods

Most readers probably automatically interpret this S.A.T. problem as follows:

How many times does the circumference of the smaller circle fit into the circumference of the second one?

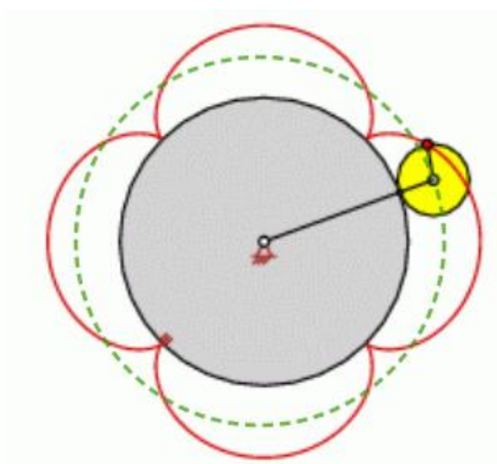
In fact, this is a completely distinct problem. Its meaning comes down to answering the question of how many rotations the smaller circle must make to trace a path equal in length to the circumference of the larger circle, i.e. when the circle rolls along a straight line. It is common knowledge that in such a case any point on the circle describes a so-called cycloid, while the center of the circle moves along a straight line (see the figure below).



The circumference of the large circle is three times the circumference of the small circle. If the small circle were to rotate among a straight line segment equal in length to the circumference of the large circle, it would make three revolutions.

In S.A.T. problem, the motion of the small circle is not in a straight line, but rather around the large circle. This revolving action around the large circle contributes an extra revolution. Let us explain it in details.

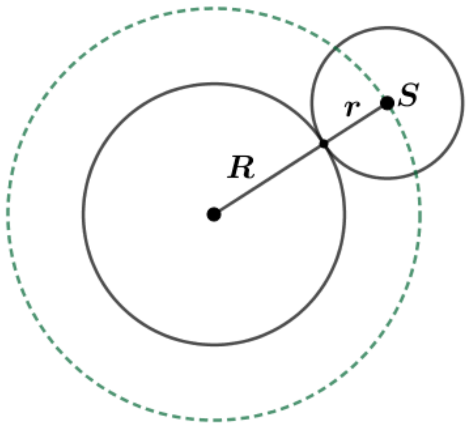
From now on we establish, that by circle B we mean the biggest one, and the smaller circle A is rolling along the B. When the circle rolls around the circumference of another, any point on the circle describes a so-called epicycloid. In such case, the center of the circle A draws a circle with a radius that is the sum of both radii, namely circles A and B (see the figure below).



Thus, to establish the number of rotations of the circle A around the circle B (to return to its starting position), we in fact need to compare the circumference of circle A with the circumference of the circle that the center of circle A will trace out as it makes those rotations (green circle on the figure above). Not the circumferences of the circles A and B, as it might seem at the beginning.

Results

We establish now the number of these revolutions, depending of the radii of the circles A and B.



Let us denote by R the radius of the circle B and by r the radius of the circle A. Thus the point S (the center of the circle A) draw a circle with circumference

$$2\,\pi\,(R+r).$$

Since the circumference of the circle A is $2\pi r$, the number of rotation is given by the ratio

$$\frac{2\,\pi\,(R+r)}{2\pi r}=\frac{R}{r}+1.$$

Thus revolving action around the large circle always contributes an one extra revolution of the circle A, comparing to revolving action along the line.

Discussion

In this section we will consider possible causes of incorrect solutions to problem described in introduction. Mistakes in mathematical problem solving stem from a wide range of cognitive, linguistic, and instructional factors. We do not consider here such factors as insufficient prior knowledge or limited opportunities for reasoning and problem solving by some students. Let us recall, that circle rotation problem comes from S.A.T. exam. If student gaps in foundational skills, struggles with basic operations, fractions, or variable manipulation – we may assume that he is not a potential candidate for the S.A.T. exam. Similarly if student has limited mental capabilities. We will therefore consider the factors that may cause an error in a candidate who has the appropriate skills and preparation.

One of the major source of errors comes from *procedural difficulties*, where students know the concept but struggle with the execution. This includes arithmetic mistakes, misordered steps, or forgetting specific rules in multi-stage problems. Procedural errors are often connected to weaknesses in working memory: when a problem requires tracking several steps at once, students may lose intermediate results or incorrectly sequence operations. These mistakes do not necessarily reflect a lack of understanding, but rather limitations in cognitive capacity or the automatization of basic skills. Ashcraft & Kirk (2001) demonstrate that working - memory overload - whether due to task complexity or anxiety - significantly increases procedural mistakes in mathematical tasks.

Another cause of the students' mistakes is struggling with *interpreting mathematical representations*, including graphs, symbols, equations, and diagrams. Duval (2006) argues that mathematical competence relies heavily on moving between different semiotic representations; when students cannot translate between verbal, algebraic, and visual forms, their problem - solving deteriorates. Misreading or misinterpreting a representation therefore becomes a direct source of incorrect solutions.

A further contributing factor involves *weak metacognitive regulation*. Students who lack the ability to plan, monitor, and evaluate their strategies are more likely to choose inefficient methods or fail to recognize errors in their reasoning. Schoenfeld (1992) highlights that successful problem solvers actively reflect on the appropriateness of their strategies.

A significant cause of mathematical mistakes *is insufficient number sense*, which limits a student's ability to judge whether a result is reasonable. Students with weak number sense often perform calculations mechanically without considering magnitude, plausibility, or alternative strategies. According to McIntosh & … (1992), deficiencies in number sense lead students to accept incorrect answers because they cannot mentally estimate or check results against intuitive expectations. Such a disorder does not necessarily have to be related to lack knowledge or the disability to think logically.

Another important factor is *overreliance on superficial heuristics*, such as keyword spotting or pattern matching, instead of analyzing the structure of the problem. When students assume that certain words automatically signal certain operations ("altogether" means add, "difference" means subtract), they often misapply procedures in contexts where such rules do not hold. Mayer (1982) notes that students who depend on shallow linguistic cues instead of building semantic representations are more prone to systematic errors in word - problem solving.

A further cause involves *misconceptions formed from instruction*, often referred to as "instructional artifacts." Sometimes teaching methods unintentionally promote incorrect generalizations - for example, using limited examples that lead students to believe all triangles are acute or that multiplication always increases a number. Smith & … (1994) explain that such faults are not random mistakes but stable mental models built through experience, which can lead students to repeat the same errors even after instruction.

And finally, mistakes in mathematical problem solving frequently arise from *conceptual misunderstandings*, in which learners fail to grasp the principles underlying procedures. Students who rely on memorized rules without deep understanding often overgeneralize or misapply methods when conditions change. Research by Hiebert & Lefevre (1986) shows that procedural knowledge without conceptual foundations leads to rigid, error - prone reasoning, especially when students encounter non-routine problems.

As it was mentioned before, candidates taking the exam were rather well - prepared. However, many had interpreted the task as if it were a completely different problem. And incorrect solution with result 3 was in fact quite reasonable and it was the most often chosen answer.

Since we ask about the number of rotations of the circle A around the circle B (to return to its starting position), it is natural to compare the circumferences of these circles. Candidates had chosen answer 3 based on some reasonable calculations. They were able to explain their reasoning. They just missed one detail. Namely, they focused on the circles sizes, with wondering, what is the essence of the rotation.

During rotation, certain points move, tracing specific curves. And the parameters of these curves have a significant impact on number of rotations. This is an aspect that most of exam takers did not take into account. Even though they understood the task and were able to interpret it correctly.

It seems that none of the causes of mathematical errors mentioned above apply here. Rather, we are dealing with inconsistencies in reasoning. In my opinion we have here a phenomenon that can be called a misconceptions in reasoning or false beliefs.

Misconceptions are persistent, systematic errors in thinking that arise when learners form incorrect mental models about mathematical ideas. These false beliefs are not simple mistakes. Rather, they are coherent internal explanations that students believe to be true and therefore apply consistently. As Smith & … (1994) note, misconceptions function as stable cognitive structures that guide reasoning, often causing learners to interpret new information incorrectly or resist conceptual change. Because these beliefs are internally logical from the learner's perspective, they can be remarkably resistant to instruction, even after the learner receives correct explanations.

One common source of false beliefs is overgeneralization of rules. Students often extend patterns that work in one context into situations where they no longer apply - such as believing that multiplication always results in a larger number or that the graph of any function must be a smooth curve. This tendency was evident in solutions to the rolling circles problem by candidates taking the S.A.T. exam. Comparing circumferences has provided the correct solution in many similar problems. Even in a nearly identical one, where only rolling along a circle is replaced by rolling along a straight line. So why wouldn't it work this time?

Misconceptions also persist because students frequently rely on intuitive reasoning that conflicts with formal mathematics**.** Research in cognitive psychology shows that people tend to apply everyday logic or perceptual intuition to mathematical problems, particularly in geometry and probability. Fischbein (1987) argues that intuitive constructs - such as "a longer side must correspond to a larger angle" - are deeply rooted and can override formal reasoning even among advanced learners.

When intuition contradicts mathematical principles, students may trust their intuitive belief more strongly than abstract definitions or proofs. The coin rotation paradox fits perfectly into this trend. Everyone assumes that if the coins are identical, there should be only one spin. Only the experiment described in the introduction forces us to consider why there isn't one, but two spins.

Fighting false beliefs is a teaching challenge. It requires more than correcting errors. Effective strategies should include cognitive conflict, multiple representations, comparison of contrasting examples, and encouraging metacognitive reflection. Without such interventions, false beliefs can remain hidden and continue to produce systematic errors across multiple mathematical domains.

Conclusion

Although the problem of coin rotation paradox has been known since the time of Aristotle, it still generates considerable interest today. In a recent paper Santos-Pereira (2025) studies the roulettes, i.e. curves that occur when one curve rolls without slipping along another, tracing the path of a fixed point. He discuss, among other, the parametric equations of cycloids and epicycloids and gives the computational implementation of those curves.

Baez & Huerta (2014) use the concept of circle rolling around the other circle to create a realization of the smallest Lie group, i.e. G_2. They use exactly the model with the radii of the circles being in the ratio $1:3$. Padyala (2019) performs a historical and mathematical review, how was the problem of quickest descent solved by Bernoulli. The brachistochrone curve turned out to be the cycloid.

In general, cycloids, epicycloids, hypocycloids are the subjects of research in many areas of science. They appear in pure mathematics, but also in engineering and technology. Flores & … (2011) use dynamic geometry software to draw the mechanical curves most used in engineering. Valk (2024) studies the cycloids using the tools of linear algebra.

They are also present in experimental area. Pimenta Ramos de Oliveira & Soltau are examine their characteristics based on video analysis in physics and Ben Abu & … (2018) use the properties of cycloids to determine the point mass in physical experiments. It is safe to say that they are some of the most intriguing objects in curve theory.

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