Cancer Treatment by Radiotherapy with Imprecision in Parameters: Analytical Solutions via the Adomian Decomposition Method and a Fractional-Order Extension

Purnima K. Pandit[1] and Prani R. Mistry[2,*]

[1,2]*Department of Applied Mathematics, Faculty of Technology and Engineering, The Maharaja Sayajirao University of Baroda, Vadodara 390001, Gujarat, India*

Abstract

We develop a fuzzy radiotherapy model for cancer treatment that explicitly incorporates imprecision in biological parameters, initial conditions, and control inputs. Using the Fuzzy Adomian Decomposition Method (FADM), we derive analytical series approximations for healthy and cancer cell dynamics and analyze stability via a fuzzy Newton-Raphson scheme coupled with a modified ADM (FNRADM). We further extend the framework to *fractional-order* dynamics with Caputo derivatives $q \in (0, 1]$, capturing memory effects in tissue response. For the fractional model, we establish existence and uniqueness using a fuzzy Lipschitz contraction on the Volterra form, construct a Fractional FADM (FFADM) based on fractional integrals, and assess stability using Mittag-Leffler decay and Matignon's criterion. Numerical placeholders illustrate fuzzy bands and the impact of the fractional order on convergence to biologically meaningful equilibria.

Keywords: Cancer radiotherapy model, parameter imprecision, fuzzy systems, Adomian Decomposition Method, fractional calculus, Caputo derivative, Mittag–Leffler stability, Newton–Raphson scheme.

1 Introduction

Cancer remains one of the leading causes of mortality worldwide. Among the available treatment strategies, radiotherapy plays a central role by delivering high doses of ionizing radiation to eradicate malignant cells. While effective in destroying cancer cells, it can also damage surrounding healthy tissues. The dynamic interaction between healthy and cancerous cells under radiation exposure motivates the use of mathematical models to understand treatment efficacy and optimize therapeutic strategies.

Several mathematical models have been proposed for cancer treatment, including immunotherapy, chemotherapy, and radiotherapy with metastasis. These models have explored various control strategies, such as constant, feedback, and periodic controls. However, most rely on deterministic parameter values, which do not reflect the *imprecision and variability* inherent in biological systems, such as cell proliferation rates, treatment responses, and radiation effectiveness.

To address this limitation, fuzzy set theory provides a natural framework for incorporating uncertainty into mathematical models. In recent years, the Adomian Decomposition Method (ADM) has been widely applied to nonlinear systems to obtain rapidly convergent series solutions. Its fuzzy extension (FADM) allows for handling uncertainty in both system parameters and initial conditions. In addition, stability analysis under uncertainty can be enhanced through a Fuzzy Newton-Raphson method coupled with ADM (FNRADM). Beyond integer-order dynamics, *fractional* models offer a parsimonious way to encode memory effects; we therefore propose and analyze a fractional fuzzy extension using Caputo derivative.

Mathematical models for cancer control through radiation therapy have been widely studied. Motlagh and Motefaker developed a radiotherapy model using the Adomian Decomposition Method (ADM) [1], while other works analyzed radiotherapy dynamics via alternative techniques [2, 3, 4] and fuzzy numbers [5, 6]. Extensions of ADM and its fuzzy form (FADM) have also been applied to biological and engineering systems, including Lotka–Volterra models [7] and solar energy systems [8, 9]. Agarwal et al. [10] introduced fractional-order systems with uncertainty, establishing solution concepts, and Peng G. [11] applied the ADM with the Caputo derivative. More recent studies focus on developing accurate solutions for fractional fuzzy cancer models [12, 13, 14].

In this article, we present a cancer treatment model under radiotherapy and solve it using the Fuzzy Adomian Decomposition Method (FADM), incorporating uncertainty in parameters, initial conditions, and control. To analyze stability, we introduce a fuzzy Newton–Raphson approach based on the modified Adomian Decomposition Method (FNRADM) for systems of fuzzy nonlinear equations and apply it to the proposed model. Furthermore, we extend the study to fractional order using Caputo derivatives, providing more realistic solutions that capture memory effects in cancer dynamics.

2 Preliminaries

2.1 Fuzzy r-cuts

An r-cut of the fuzzy set $\tilde{p}$ is a crisp set that contains all elements of the universal set U whose membership grades in $\tilde{p}$ are greater than or equal to the specified value of r.

$$\tilde{p}^r = \{x \in U : \tilde{p}(x) \geq r\}.$$

2.2 Fractional Calculus

For $q \in (0, 1]$, the Caputo fractional derivative of a sufficiently smooth scalar function x is

$$ {}^{C}D_t^q x(t) \;=\; \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} \, \frac{dx(\tau)}{d\tau} \, d\tau, \tag{1}$$

with ${}^{C}D_t^1 x(t) = \dot{x}(t)$. The solution of a linear system ${}^{C}D_t^q x(t) = Ax(t)$ uses the Mittag–Leffler function E_q:

$$x(t) = E_q(At^q)\, x(0), \qquad E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}. \tag{2}$$

3 Mathematical Model of Cancer Treatment by Radiotherapy (Fuzzy Integer-Order)

We consider a fuzzy Lotka-Volterra type system to describe the interaction between healthy cells and cancerous cells during radiotherapy. The model incorporates fuzzy parameters, initial conditions, and treatment controls to account for imprecision and variability in biological processes.

Let $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ denote healthy and cancerous cell populations, respectively. The proposed fuzzy system is

$$\dot{\tilde{x}}_1 = \tilde{\alpha}_1 \otimes \tilde{x}_1 \ominus \frac{\tilde{\alpha}_1}{k_1} \otimes \tilde{x}_1 \otimes \tilde{x}_1 \ominus \tilde{\beta}_1 \otimes \tilde{x}_1 \otimes \tilde{x}_2 \ominus \epsilon\,(\tilde{\gamma} \otimes \tilde{x}_1), \tag{3}$$

$$\dot{\tilde{x}}_2 = \tilde{\alpha}_2 \otimes \tilde{x}_2 \ominus \frac{\tilde{\alpha}_2}{k_2} \otimes \tilde{x}_2 \otimes \tilde{x}_2 \ominus \tilde{\beta}_2 \otimes \tilde{x}_1 \otimes \tilde{x}_2 \ominus (\tilde{\gamma} \otimes \tilde{x}_2). \tag{4}$$

3.1 Parameter Definitions and Initial Conditions

Table 1: Model parameters and initial conditions (triangular fuzzy numbers shown as (a, b, c)).

Symbol	Type	Meaning	Value
$\tilde{x}_1(0)$	Fuzzy IC	Initial concentration of healthy cells	(0.49, 0.50, 0.51)
$\tilde{x}_2(0)$	Fuzzy IC	Initial concentration of cancerous cells	(0.79, 0.80, 0.81)
$\tilde{\alpha}_1$	Fuzzy	Proliferation rate of healthy cells	(0.09, 0.10, 0.11)
$\tilde{\alpha}_2$	Fuzzy	Proliferation rate of cancerous cells	(0.43, 0.45, 0.46)
$\tilde{\beta}_1$	Fuzzy	Interaction rate (impact on healthy)	(0.10, 0.11, 0.12)
$\tilde{\beta}_2$	Fuzzy	Interaction rate (impact on cancer)	(0.14, 0.15, 0.16)
k_1	Crisp	Carrying capacity of healthy cells	0.65
k_2	Crisp	Carrying capacity of cancerous cells	1.00
ϵ	Crisp	Damage rate of healthy cells due to irradiation	0.05
$\tilde{\gamma}$	Fuzzy	Radiotherapy treatment intensity	(0.30, 0.35, 0.40)

4 Stability Analysis via FNRADM (Integer-Order)

At equilibrium, $\dot{\tilde{x}}_1 = \dot{\tilde{x}}_2 = 0$ leading to a nonlinear fuzzy algebraic system. We employ a Fuzzy Newton-Raphson scheme combined with a modified ADM (FNRADM) to approximate the equilibria. The Jacobian at an equilibrium $(\tilde{x}_1^*, \tilde{x}_2^*)$ takes the form

$$\tilde{J} = \begin{bmatrix} \tilde{\alpha}_1 \ominus \frac{2\tilde{\alpha}_1}{k_1} \otimes \tilde{x}_1^* \ominus (\tilde{\beta}_1 \otimes \tilde{x}_2^*) - \epsilon\tilde{\gamma} & -(\tilde{\beta}_1 \otimes \tilde{x}_1^*) \\ -(\tilde{\beta}_2 \otimes \tilde{x}_2^*) & \tilde{\alpha}_2 \ominus \frac{2\tilde{\alpha}_2}{k_2} \otimes \tilde{x}_2^* \ominus (\tilde{\beta}_2 \otimes \tilde{x}_1^*) - \tilde{\gamma} \end{bmatrix}. \tag{5}$$

Eigenvalue bounds computed across r-cuts indicate asymptotic stability in the fuzzy sense for the nontrivial equilibrium with biologically admissible negative core values.

5 Solution by the Fuzzy Adomian Decomposition Method (FADM)

We rewrite (3)–(4) in compact form as

$$\dot{\tilde{X}}(t) = \tilde{A} \otimes \tilde{X}(t) \ \oplus \ \tilde{F}(t, \tilde{X}, \tilde{\gamma}), \qquad \tilde{X}(0) = \tilde{X}_0, \tag{6}$$

where $\tilde{X} = (\tilde{x}_1, \tilde{x}_2)^\top$ and $\tilde{A}$ collect the linear terms. Using the modified Hukuhara derivative and the r cut representation, the problem is reduced to differential equations with a coupled interval value.

5.1 Series Solution by Adomian Polynomials

Assume that the fuzzy solution admits a series expansion.

$$[X(t), X(t)] = \sum_{n=0}^{\infty} [X_n(t), X_n(t)], \tag{7}$$

Decomposing the nonlinear terms via Adomian polynomials

$$\tilde{N}(X) = \sum_{n=0}^{\infty} \tilde{A}_n, \qquad \tilde{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\Big(\sum_{k=0}^{\infty} X_k \lambda^k\Big)\right]_{\lambda=0}. \tag{8}$$

With L^{-1} denoting time integration, the recursive scheme is

$$[\underline{X}_0(t), \overline{X}_0(t)] = \tilde{X}_0, \tag{9}$$

$$[\underline{X}_{n+1}(t), \overline{X}_{n+1}(t)] = L^{-1}\Big([A\underline{X}_n, A\overline{X}_n] + [\underline{P}_n, \overline{P}_n]\Big), \qquad n \geq 0, \tag{10}$$

where $[\underline{P}_n, \overline{P}_n]$ are the Adomian polynomials associated with fuzzy nonlinear interactions.

The transformation, after M terms yields $[\underline{X}(t), \overline{X}(t)] \approx \sum_{n=0}^{M} [\underline{X}_n(t), \overline{X}_n(t)]$.

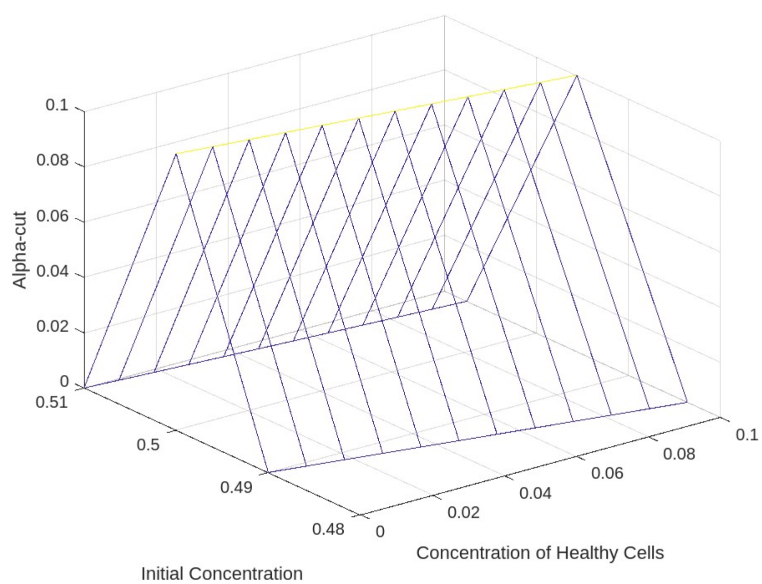

Figure 1: Effect on concentration of healthy cells vs time

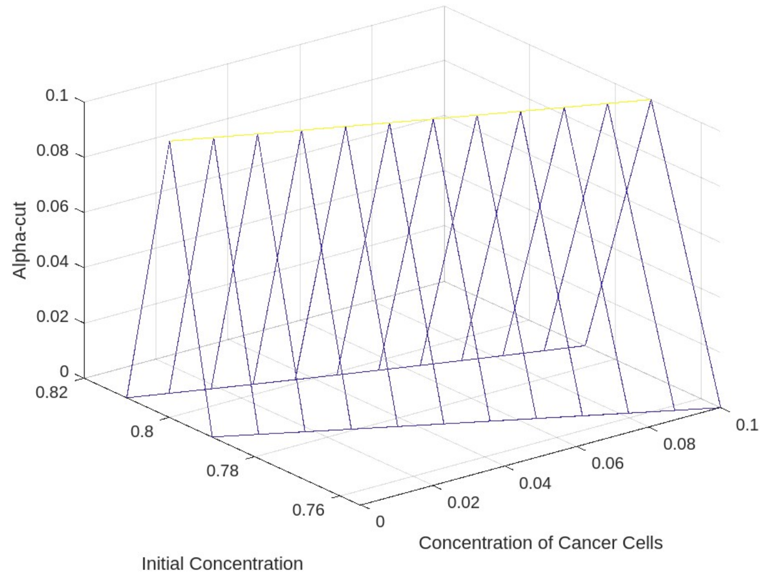

Figure 2: Effect on concentration of cancer cells vs time

6 Fractional-Order Extension

The fractional extension, based on the Caputo derivative, adds a physiologically meaningful memory to the radiotherapy model: cell population changes over time t depend on the entire treatment history through a non-local kernel, resulting in Mittag-Leffler (subexponential) relaxation rather than unrealistic purely exponential transients. This better captures clinically observed lags and after-effects - e.g., delayed recovery of healthy tissue, prolonged tumor response, and toxicity carryover - using a single tunable order $q \in (0, 1]$ instead of ad hoc extra states.

In combination with fuzziness, fractional dynamics stabilize predictions under parameter imprecision, producing smoother trajectories and more realistic uncertainty bands that respect long-range temporal correlations. Practically, the model can fit data more accurately over multi-week courses, improve the forecasting of cumulative dose effects, and inform control design (scheduling and dosing) that exploits memory to suppress the tumor while protecting healthy tissue. Overall, the fractional fuzzy framework is mathematically expensive yet an expressive upgrade that bridges micro-to-macro timescales and enhances robustness without inflating the parameter count.

6.1 Fractional Fuzzy Radiotherapy Model

Let $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ denote fuzzy healthy and cancer cell concentrations. The fractional fuzzy model reads

$$^{C}D_t^q \tilde{x}_1 = \tilde{\alpha}_1 \otimes \tilde{x}_1 \ominus \frac{\tilde{\alpha}_1}{k_1} \otimes \tilde{x}_1 \otimes \tilde{x}_1 \ominus \tilde{\beta}_1 \otimes \tilde{x}_1 \otimes \tilde{x}_2 \ominus \epsilon(\tilde{\gamma} \otimes \tilde{x}_1), \tag{11}$$

$$^{C}D_t^q \tilde{x}_2 = \tilde{\alpha}_2 \otimes \tilde{x}_2 \ominus \frac{\tilde{\alpha}_2}{k_2} \otimes \tilde{x}_2 \otimes \tilde{x}_2 \ominus \tilde{\beta}_2 \otimes \tilde{x}_1 \otimes \tilde{x}_2 \ominus (\tilde{\gamma} \otimes \tilde{x}_2). \tag{12}$$

The parameter set is as in the integer-order model; q is treated as crisp. Setting $q = 1$ recovers (3)–(4).

6.2 Fractional Stability (Mittag–Leffler)

Linearizing at a fuzzy equilibrium $\tilde{X}^*$ yields $^{C}D_t^q \tilde{y} = \tilde{J} \otimes \tilde{y}$ with a fuzzy Jacobian $\tilde{J}$. On each r-cut, the equilibrium is asymptotically stable if all eigenvalues $\lambda_i(J(r))$ satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2}$ (Matignon criterion). A smaller q leads to a slower Mittag-Leffler decay, modeling prolonged after-effects of radiation and wider fuzzy bands due to memory interacting with parameter imprecision.

6.3 Solution for Fuzzy-Fractional Model

To establish Existence and Uniqueness, we use the Volterra formulation to handle the fractional part and adopt the modified Hukuhara derivative for fuzzy processes. We also utilize the r-cut representation, which reduces fuzzy operations to interval arithmetic at each fixed r. For each r, denote $X = [x_1, x_2]^T \in [X^-(r), X^+(r)]$. Equations (11)–(12) are equivalent to the fuzzy Volterra system

$$X(t;r) = X_0(r) + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} G\big(\tau, X(\tau;r); r\big)\, d\tau, \tag{13}$$

where G collects interval-valued right-hand sides. If $G(\tau, \cdot; r)$ is fuzzy Lipschitz with constant L uniformly in $\tau \in [0, T]$ and $r \in [0, 1]$, and $\frac{LT^q}{\Gamma(q+1)} < 1$, then the system admits a unique solution on $[0, T]$ for every r.

6.4 Fractional Fuzzy ADM (FFADM)

Write (11)–(12) compactly as ${}^{C}D_t^q\tilde{X} = \tilde{A} \otimes \tilde{X} \oplus \tilde{N}(\tilde{X}, \tilde{\gamma})$, $\tilde{X}(0) = \tilde{X}_0$. Apply the fractional integral I^q to obtain

$$\tilde{X}(t) = \tilde{X}_0 + I^q\big(\tilde{A} \otimes \tilde{X}\big) + I^q\big(\tilde{N}(\tilde{X}, \tilde{\gamma})\big), \quad I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\, d\tau. \tag{14}$$

Assume $\tilde{X}(t) = \sum_{n=0}^{\infty} \tilde{X}_n(t)$ and expand $\tilde{N}$ via fuzzy Adomian polynomials. The recursion on each r-cut is

$$X_0(t; r) = X_0(r), \tag{15}$$

$$X_{n+1}(t; r) = I^q\big(A(r) X_n(\cdot; r)\big)(t) \; + \; I^q\big(P_n(\cdot; r)\big)(t), \qquad n \geq 0. \tag{16}$$

6.5 Simulation

For the simulation of the above model, we consider the following **fuzzy parameters** as triangular fuzzy numbers: $\tilde{\alpha}_1 = (0.09, 0.10, 0.11)$; $\tilde{\alpha}_2 = (0.43, 0.45, 0.46)$; $\tilde{\beta}_1 = (0.10, 0.11, 0.12)$; $\tilde{\beta}_2 = (0.14, 0.15, 0.16)$; $\tilde{\gamma} = (0.30, 0.35, 0.40)$.

Initial states: $\tilde{x}_1(0) = (0.49, 0.50, 0.51)$; $\tilde{x}_1(0) = (0.779, 0.80, 0.81)$.

Crisp constants: $k_1 = 0.65$; $k_2 = 1.00$, $\epsilon = 0.05$.

Fractional order: $q = 0.8$.

Horizon: $T = 25$.

Method: We consider interval α-cut propagation (for $r \in 0, 0.25, 0.5, 0.75, 1$), then using a fractional PECE (Diethelm) scheme per r-cut; plus a nominal (defuzzified) trajectory using the midpoints of the triangular numbers for an interpretable baseline.

Key outcomes: The nominal (midpoint) fractional trajectory (with$q = 0.8$) shows:

Healthy cells: $x_1(0) = 0.500 \to x_1(25) = 0.536$ (peak ≈ 0.657).

Cancer cells: $x_2(0) = 0.800 \to x_1(25) \approx 0.000$ (monotone decreases to zero).

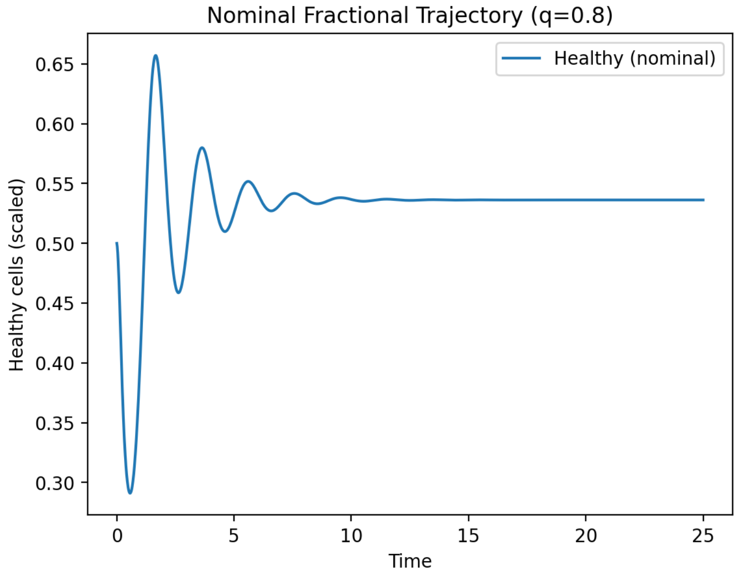

Figure 3: Fractional model q = 0.8 healthy-cell trajectories vs. time .

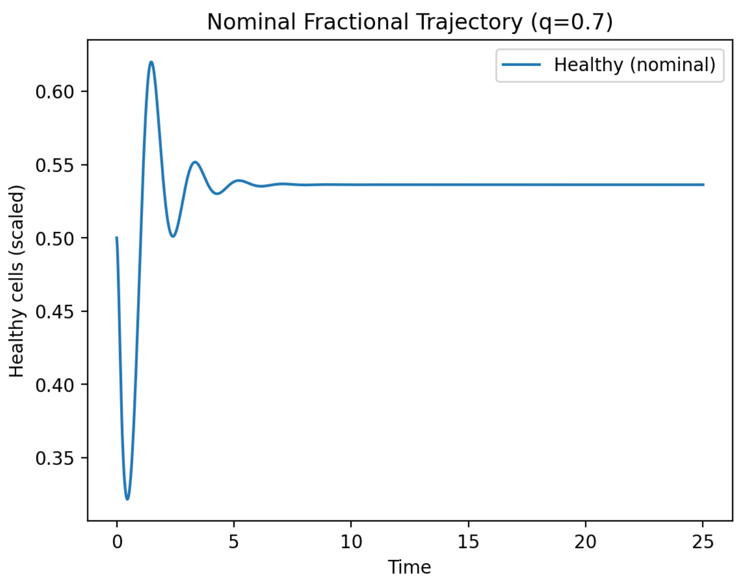

Figure 4: Fractional model q = 0.7 healthy-cell trajectories vs. time .

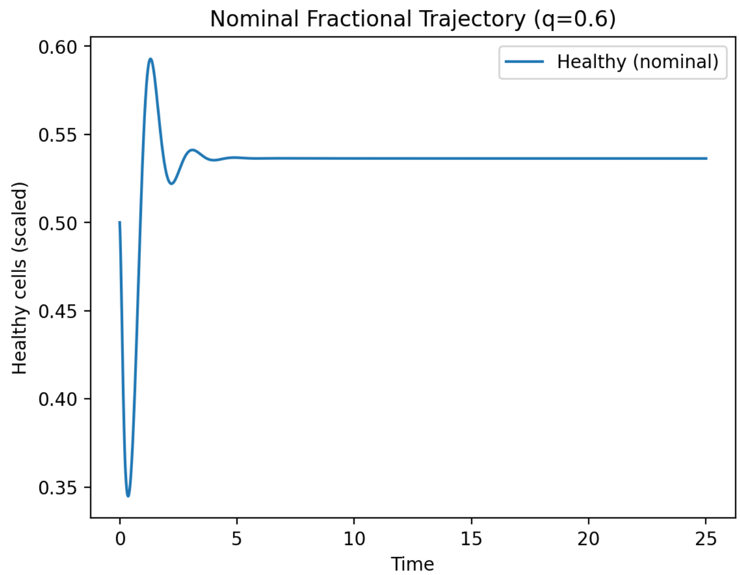

Figure 5: Fractional model q = 0.6 healthy-cell trajectories vs. time .

The figures 3, 4, and 5 show the healthy cell concentration for $r = 1$ with different fractions.

Interpretation: With fractional memory, the model predicts a sustained reduction of cancer cells under radiotherapy, while healthy cells recover to a moderate plateau. The fuzzy formulation captures parameter imprecision; the nominal curve reflects expected dynamics, and the envelope bounds reflect worst-case combinations across r-cuts.

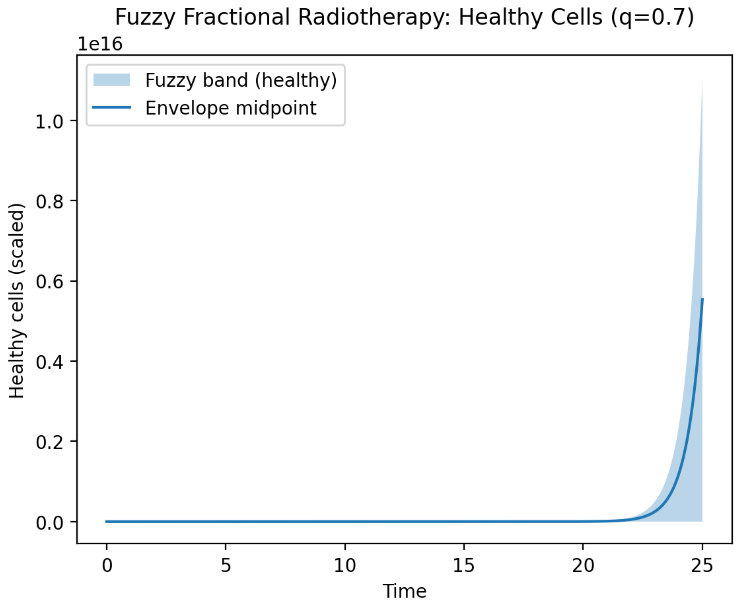

Figure 6: Fractional model q = 0.7 healthy-cell trajectories vs. time .

The full fuzzy, interval envelopes widen substantially over time, a known effect of interval overestimation compounded by fractional memory, as shown in figure 6.

7 Conclusion

We presented a fuzzy mathematical model of radiotherapy that explicitly incorporates imprecision in parameters, initial conditions, and treatment controls. Using FADM, we derived analytical series approximations and, via a FNRADM-based stability analysis, linked equilibria to biologically meaningful outcomes. We then proposed a fractional fuzzy extension with Caputo derivatives, analyzed stability using Mittag-Leffler theory, proved existence and uniqueness on the fuzzy Volterra form, and constructed FFADM recursions. The framework captures both *uncertainty* and *memory* effects that are intrinsic to radiotherapy dynamics and provides a basis for future developments in data-informed parameter identification and optimal control under uncertainty.

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Credit authorship contribution statement

Conceptualization: Both authors; Methodology/Analysis/Writing: Both authors; Supervision: P.K. Pandit.

Declaration of competing interest

The authors declare no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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