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Strong Uniform Consistency rates of some Conditional Nonparametric Functions with Functional α-Mixing Data

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Abstract In this paper, we investigate a nonparametric local linear estimation of some conditional functions of a scalar response variable given a functional random covariate. Firstly, we establish both pointwise and uniform almost-complete convergence of the conditional distribution, when the sample is an α-mixing sequence. Then, we deduce the uniform almost-complete convergence of the conditional quantile estimator. Furthermore, we present the successive derivatives of conditional distribution estimators and its asymptotic properties. Finally, a simulation study is carried out to show the performance of a studied estimator with respect to the kernel method.

Keywords Functional data · Nonparametric estimation · Rate of Convergence · Uniform almost-complete convergence.

Mathematics Subject Classification (2010) 62G05 · 62G20 · 62G08

1 Introduction

Functional data is an important subject in modern nonparametric statistics. During the last years, many works have been devoted to theoretical results and applied studies on models involving functional data such as the conditional mode, the conditional median and the conditional quantile.

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In many situations we have to deal with dependent functional datasets. One of the most popular examples come from the study of a strong-mixing through the functional approach proposed by [7] for nonparametric conditional models, they established their pointwise almost-complete convergence. [10,1] obtained the asymptotic normality of the conditional quantile and conditional density of the nonparametric kernel estimators. On the other hand, [5] and [12] established the asymptotic properties for the functional locally modeled conditional density and generalized regression function.

The first goal of the present paper is to study, under some conditions, the rate of the pointwise almost-complete convergence for dependent data of the local linear estimator of the conditional distribution which introduced in [14] (see section 2).

Other works have been conducted on dependent and functional data case. We refer to [3] where the authors established the asymptotic normality results under selected α-mixing stationary data of the local linear estimator of conditional cumulative distribution. The almost-complete consistency with rates of the local linear estimator of the conditional density is stydied in [2] and the asymptotic normality of the conditional mode estimator is obtained by [4].

Notice that the interest of the uniform consistency comes mainly from the pointwise performance of all estimators and it is not a direct extension of the previous pointwise result. In the independent case, [8] proved the uniform convergence for the kernel estimators, while [6] and [14] proved it for the local linear estimators. In the dependent case, we cite [11] for the generalized regression function. Based on the above explanation, our second goal consists in establishing the uniform convergence of the local linear estimator of the conditional distribution and the conditional quantile (see section 3). In section 4, we give the estimators of the successive derivatives of conditional distribution which implies the conditional density and conditional hazard and we studied their asymptotic properties. In section 5, a simulation study is used to illustrate the performance of the studied conditional median estimator with respect to the kernel method. The last section is devoted to the proofs of some theoretical results.

Throughout this paper the following notations will be adopted. Let $\mathscr{F}$ and $S_{\mathbb{R}}$ denote respectively an infinite-dimensional space equipped with a semimetric d and a fixed compact subset of $\mathbb{R}$, X is a random variable valued in $\mathscr{F}$, for any $x \in \mathscr{F}$, $h > 0$, $B(x,h) := \{y \in \mathscr{F} / \, |\delta(x,y)| \leq h\}$ denotes a closed ball in $\mathscr{F}$ of center x and radius h. We also define $\Phi_x(r_1,r_2) := P(r_1 \leq |\delta(x,y)| \leq r_2)$, where r_1 and r_2 are two real numbers.

For easy reference, the following definitions need to be recalled.

Definition 1 Let $\{Z_i, i = 1,2,...\}$ be a strictly stationary sequence of random variables, $F_i^k(Z)$ denotes the σ-algebra generated by $\{Z_j, i \leq j \leq k\}$. Given a positive integer n, set

$$\alpha(n) = sup\{|P(A \cap B) - P(A)P(B)| : A \in F_1^k(Z) \text{ and } B \in F_{k+n}^{\infty}(Z), k \in \mathbb{N}^*\}.$$

The sequence is said to be α-mixing (strongly mixing) if the mixing coefficient $\alpha(n) \to 0$ as $n \to \infty$.

Many processes do satisfy the strong mixing property, such that ARMA, ARCH and GARCH processes, see for example [13] for more details.

Definition 2 Let $(Z_n)_{n\in\mathbb{N}^*}$ be a sequence of real random variables (r.r.v.). We say that $(Z_n)_{n\in\mathbb{N}^*}$ converges almost completely to some r.r.v. Z, and we note $Z_n \overset{a.co.}{\longmapsto} Z$, if and only if

$$\forall \varepsilon > 0, \ \sum_{n=1}^{\infty} P(|Z_n - Z| > \varepsilon) < \infty.$$

Moreover, let $(u_n)_{n\in\mathbb{N}^*}$ be a sequence of positive real numbers going to zero; we say that the rate of the almost complete convergence of $(Z_n)_{n\in\mathbb{N}^*}$to Z is of order (u_n) and we note $Z_n - Z = O_{a.co.}(u_n)$, if and only if

$$\exists \varepsilon_0 > 0, \ \sum_{n=1}^{\infty} P(|Z_n - Z| > \varepsilon_0 u_n) < \infty.$$

We study this convergence because it is stronger than the almost-sure one (see Borel Cantelli lemma).

2 Model and pointwise almost-complete convergence

Let us consider n pairs of random variables $(X_i, Y_i)_{i=1,\dots,n}$ identically distributed as the pair (X,Y) which is valued in $\mathscr{F} \times \mathbb{R}$.
We assume that there exists a regular version of the conditional distribution of Y given X, denoted by F_Y^x for a fixed object $x \in \mathscr{F}$, where $F_Y^x(y) = P(Y \leq y \mid X = x)$.
Following [14], the local linear estimator $\widehat{F}_Y^x(y)$ of $F_Y^x(y)$ is given by

$$\widehat{F}_Y^x(y) = \frac{\sum_{i,j=1}^{n} W_{ij}(x) H(g^{-1}(y - Y_j))}{\sum_{i,j=1}^{n} W_{ij}(x)}, \tag{1}$$

with the convention $0/0 = 0$ and

$$W_{ij}(x) = \beta(X_i,x)\,(\beta(X_i,x) - \beta(X_j,x))\,K(h^{-1} \mid delta(X_i,x)|)K(h^{-1}\left|\delta(X_j,x)\right|),$$

where $\beta(.,.)$ is a known function from $\mathscr{F} \times \mathscr{F}$ into $\mathbb{R}$ such that, $\forall \xi \in \mathscr{F}$, $\beta(\xi,\xi) = 0$ and $\delta(.,.)$ is such that $d(.,.) = |\delta(.,.)|$. The functions K and H are kernels (with $H(u) = \int_{-\infty}^{u} H_0(v)dv$ where H_0 is a kernel of type 0 and $h = h_{K,n}$ (resp. $g = g_{H,n}$) is a sequence of strictly positive real numbers which plays a smoothing parameter role.

Among the important applications of the conditional distribution estimation is the estimation of the conditional quantile of order α which defined by

$$q_\alpha(x) = \inf\{y \in \mathbb{R}, F^x(y) \geqslant \alpha\}.$$

A natural estimator of $q_\alpha(x)$ is

$$\widehat{q}_\alpha(x) = \inf\{y \in \mathbb{R}, \widehat{F}^x(y) \geqslant \alpha\}.$$

Remark that $q_{1/2}(x)$ is the so called conditional median.

Now we are in position to state the pointwise almost-complete convergence of the local linear estimator $\widehat{F}_Y^x(y)$, for a fixed point x in $\mathscr{F}$. For this purpose, we need the following conditions

(H1) for any $h > 0$, $\Phi_x(h) := \Phi_x(0,h) > 0$.

(H2) The conditional distribution F_Y^x satisfies for some strictly positive constants b_1,b_2 and for all $(y_1,y_2) \in S_\mathbb{R} \times B(y_1,g)$ and all $x_1 \in B(y_1,h)$, $|F_Y^{x_1}(y_1) - F_Y^{x_2}(y_2)| \leq C_x(|\delta(x_1,x_2)|^{b_1} + |y_1 - y_2|^{b_2})$, where C_x is a positive constant depending on x.

(H3) The function $\beta(.,.)$ is such that: $\exists 0 < M_1 < M_2, \forall x' \in \mathscr{F}, M_1|\delta(x,x')| \leq |\beta(x,x')| \leq M_2|\delta(x,x')|$.

(H4) The kernel K is a positive and differentiable function on its support $[0,1]$ and $\exists C, C'$ such that $0 < C1_{[0,1]}(u) \leq K(u) \leq C'1_{[0,1]}(u) < \infty$.

(H5) The sequence (X_i,Y_i) is an α-mixing sequence with coefficient $\alpha(n)$, moreover (H5a) and (H5b) are satisfied, where

(H5a) $\exists\, C > 0, \exists a > 4 + 2\xi, \forall n \in \mathbb{N}; \alpha(n) \leq Cn^{-a}$, where ξ is defined in (H7),

(H5b) $\exists\, C, C' > 0$ such that: $C'\,[\Phi_x(h)]^{a/(a-1)} < \psi_x(h) \leq C\,[\Phi_x(h)]^{a/(a-1)}$, where $\psi_x(h) := \psi_x(0,h)$ and $\psi_x(h_1,h_2) := P\,(h_1 \leq |\delta(X_1,x)| \leq h_2, 0 \leq |\delta(X_2,x)| \leq h_2)$.

(H6) The bandwidth h satisfies

$$\exists n_0 \in \mathbb{N}, \forall n > n_0, \frac{1}{\psi_x(h)} \int_0^1 \psi_x(zh,h) \frac{d}{dz}\left(z^2 K(z)\right) dz > C > 0$$

and

$$h^2 \int_{B(x,h)} \int_{B(x,h)} \beta(u,x)\beta(\vartheta,x) dP_{(X_1,X_2)}(u,\vartheta)$$
$$= o\left(\int_{B(x,h)} \int_{B(x,h)} \beta^2(u,x)\beta^2(\vartheta,x) dP_{(X_1,X_2)}(u,\vartheta)\right),$$

where $dP_{(X_1,X_2)}$ is the joint distribution of (X_1,X_2).

(H7) The bandwidths h and g satisfy

$$\lim_{n\to\infty} h = 0,\ \lim_{n\to\infty} g = 0,\ for\ some\ \ \xi > 0,\ \lim_{n\to\infty} n^\xi g = 0$$

and

$$\exists\, C_1 > 0\,,\ 0 < \eta_0 < \frac{a-4-2\xi}{a+1}\ \ such\ that\ \ C_1 n^{\frac{3-a}{a+1}+\eta_0+\frac{2\xi+1}{a+1}} \leq \Phi_x(h).$$

Our hypotheses are very standard for this kind of model. Assumptions (H1)–(H3) have been assumed in the independent case (see [14]). (H4) is a technical assumption and it is imposed only for the brevity of proofs. (H5a) means that (X_i,Y_i) is arithmetically mixing which is a standard choice for the mixing coefficient in time series. (H6) is a same condition of (H6) in [11]. The choice of the bandwidths h and g are given by (H7) which implies that $\lim_{n\to\infty} n\Phi_x(h)/\ln n \to \infty$ as $n \to \infty$.

Let's state the pointwise almost-complete convergence of $\widehat{F}_Y^x$.

Theorem 1 *Assume that the assumptions (H1)–(H7) are satisfied.*

$$\sup_{y\in S_{\mathbb{R}}} |\widehat{F}_Y^x(y) - F_Y^x(y)| = O\left(h^{b_1} + g^{b_2}\right) + O_{a.co.}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

Similary to [14], the proof of the theorem 1 is based on the following decomposition given by: $\forall y \in S_{\mathbb{R}}$,

$$\widehat{F}_Y^x(y) - F_Y^x(y) = \frac{1}{\widehat{F}_D^x}(\widehat{F}_N^x(y) - E(\widehat{F}_N^x(y))) - (F_Y^x(y) - E(\widehat{F}_N^x(y)) - \frac{F_Y^x(y)}{\widehat{F}_D^x}(\widehat{F}_D^x - 1), \quad (2)$$

where

$$\widehat{F}_N^x(y) = \frac{1}{n(n-1)E\left[W_{12}(x)\right]}\sum_{i\neq j} W_{ij}(x)H(g^{-1}(y-Y_j))$$

and

$$\widehat{F}_D^x = \frac{1}{n(n-1)E\left[W_{12}(x)\right]}\sum_{i\neq j} W_{ij}(x)$$

and of the following lemmas for which the proofs are relegated to the Appendix.

Lemma 1 *Under assumptions (H1)–(H6), we have*

$$\sup_{y\in S_{\mathbb{R}}} \left|F_Y^x(y) - E[\widehat{F}_N^x(y)]\right| = O\left(h^{b_1}\right) + O\left(g^{b_2}\right).$$

Lemma 2 *Assume that hypotheses of Theorem 1 hold, then*

$$\sup_{y\in S_{\mathbb{R}}} \left|\widehat{F}_N^x(y) - E[\widehat{F}_N^x(y)]\right| = O_{a.co.}\left(\sqrt{\frac{\ln n}{n\,\Phi_x(h)}}\right).$$

Lemma 3 *(see [11]) If assumptions (H1),(H3)–(H7) are satisfied, we get*

$$\widehat{F}_D^x - 1 = O_{a.co.}\left(\sqrt{\frac{\ln n}{n\,\Phi_x(h)}}\right) \text{ and } \sum_{i=1}^{\infty} P\left(F_D^x < \frac{1}{2}\right) < \infty.$$

3 Uniform almost-complete convergence

Notice that, unlike in the multivariate case, the uniform consistency is not a standard extension of the pointwise one. Thus, we need additional tools and topological conditions (for more details and examples, see [9] and [8]). More precisely, we study the uniform almost-complete convergence of $\widehat{F}_Y^x$ and $q_\alpha(x)$ on some subset $S_{\mathscr{F}}$ of $\mathscr{F}$, satisfing $S_{\mathscr{F}} \subset \bigcup_{k=1}^{N_{r_n}(S_{\mathscr{F}})} B(x_k, r_n)$, where for all $k \in \{1, \dots, N_{r_n}(S_{\mathscr{F}})\}$, $x_k \in S_{\mathscr{F}}$ and (r_n) is a sequence of positive real numbers.

3.1 The estimator $\widehat{F}_Y^x$

This subsection is devoted to the uniform version of Theorem 2. For this goal, we need the following assumptions

(U1) There exist a differentiable function Φ and strictly positive constants C, C_1 and C_2 such that

$$\forall x \in S_{\mathscr{F}}, \forall h > 0;\ 0 < C_1\Phi(h) \leq \Phi_x(h) \leq C_2\Phi(h) < \infty$$
$$and\ \exists \eta_0 > 0, \forall \eta < \eta_0, \Phi'(\eta) < C,$$

where Φ' denotes the first derivative of Φ with $\Phi(0) = 0$.

(U2) The conditional distribution F_Y^x satisfies for some strictly positive constants C, b_1 and b_2 and for all (x_1, x_2) in $S_{\mathscr{F}} \times B(x_1, h)$ and (y_1, y_2) in $S_{\mathbb{R}} \times B(y_1, g)$

$$|F_Y^{x_1}(y_1) - F_Y^{x_2}(y_2)| \leq C\left(|\delta(x_1, x_2)|^{b_1} + |y_1 - y_2|^{b_2}\right).$$

(U3) The function $\beta(.,.)$ satisfies (H3) uniformly on x and the following Lipschitz's condition

$$\exists C > 0, \forall x_1 \in S_{\mathscr{F}}, x_2 \in S_{\mathscr{F}}, x \in \mathscr{F}, |\beta(x, x_1) - \beta(x, x_2)| \leq C|\delta(x_1, x_2)|.$$

(U4) The kernel K fulfills (H4) and is Lipschitzian on $[0, 1]$.

(U5) The sequence (X_i, Y_i) is an α-mixing sequence with coefficient $\alpha(n)$, moreover (U5a) and (U5b) are satisfied, where

(U5a) $\exists\, C > 0, \exists a > 6 + 2\xi, \forall n \in \mathbb{N}; \alpha(n) \leq Cn^{-a}$, where ξ is defined in (H7),

(U5b) $\exists\, C_1 > 0,\ C_2 > 0$ such that $\forall\, x_1 \in S_{\mathscr{F}}, \forall\, x_2 \in S_{\mathscr{F}}$,
$0 < C_1\,[\Phi(h)]^{a/(a-1)} \leq P\,[(X_1, X_2) \in B(x_1, h) \times B(x_2, h)] \leq C_2\,[\Phi(h)]^{a/(a-1)}$.

(U6) The hypothesis (H6) is satisfied uniformly on $x \in S_{\mathscr{F}}$.

(U7) The bandwidths h and g satisfy

$$\lim_{n\to\infty} h = 0,\ \lim_{n\to\infty} g = 0,\ for\ some\ \ \xi > 0,\ \lim_{n\to\infty} n^{\xi} g = 0,$$

$$\exists\, C_1 > 0\,,\ 0 < \eta_0 < \frac{a - 6 - 2\xi}{a + 1}\ \ such\, that\ \ C_1 n^{\frac{3-a}{a+1} + \eta_0 + \frac{2\xi+3}{a+1}} \leq \Phi_x(h)$$

and for $r_n = O\left(\frac{\ln n}{n}\right)$, the function $\psi_{S_{\mathscr{F}}}$ satisfies for n large enough $\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right) \sim C\ln n$.

Roughly speaking, these hypotheses are uniform version of the assumed conditions in the pointwise case and have already been used in the literature. The last condition on entropie in (U7) is satisfied in some common cases (see [11]) and leads to find again the same rate as in the pointwise case but uniformly on x.
Our result is as follows

Theorem 2 *Under assumptions (U1)–(U7), we have*

$$\sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} |\widehat{F}_Y^x(y) - F_Y^x(y)| = O(h^{b_1}) + O(g^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right).$$

It is easy to see that the proof of Theorem 2 is a direct consequence of the decomposition (2) and of the following lemmas for which the proofs are relegated to the Appendix.

Lemma 4 *Assume that hypotheses (U1), (U2) and (U4) hold, then*

$$\sup_{x\in S_{\mathscr{F}}}\sup_{y\in S_{\mathbb{R}}}|F_Y^x(y)-E[\widehat{F}_N^x(y)]|=O(h^{b_1})+O(g^{b_2}).$$

Lemma 5 *Under assumptions of Theorem 2, we obtain*

$$\sup_{x\in S_{\mathscr{F}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{F}_N^x(y)-E[\widehat{F}_N^x(y)]|=O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right).$$

Lemma 6 *(see [11]) If assumptions (U1),(U3), (U4)–(U7) are satisfied, we get*

$$\sup_{x\in S_{\mathscr{F}}}|\widehat{F}_D^x-1|=O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right)\quad \textit{and}\quad \sum_{n=1}^{\infty}P\left(\inf_{x\in S_{\mathscr{F}}}\widehat{F}_D^x<\frac{1}{2}\right)<\infty.$$

3.2 A conditional quantile estimator

To investigate the uniform consistency of the conditional quantile estimator, we need the additional following conditions used for example in [14].

(U8) $\forall\varepsilon>0$, $\exists\xi>0$ such that for any function g_α from $S_{\mathscr{F}}$ into $[q_\alpha(x)-\delta,q_\alpha(x)+\delta]$ we have

$$\sup_{x\in S_{\mathscr{F}}}|q_\alpha(x)-g_\alpha(x)|\geqslant\varepsilon\quad\Rightarrow\quad\sup_{x\in S_{\mathscr{F}}}|F^x(q_\alpha(x))-F^x(g_\alpha(x))|\geqslant\xi.$$

(U9) $\exists j>1, \forall x\in S_{\mathscr{F}}$, F^x is j-times continuously differentiable on $]q_\alpha(x)-\delta,q_\alpha(x)+\delta[$ with respect to y and satisfies $F^{x(l)}(q_\alpha(x))=0$ if $0\le l<j$, $F^{x(j)}(q_\alpha(x))>C>0$ and $F^{x(j)}$ is uniformly continuous on $[q_\alpha(x)-\delta,q_\alpha(x)+\delta]$ where $F^{x(l)}$ stands for the l^{th} order derivative of F^x .

A known method can be applied to derive the following result from Proposition 1, see for example the proof of Corollary 3.1 in [14].

Proposition 1 *Under the hypotheses of Theorem 2 and if (U8) and (U9) are satisfied, we have*

$$\sup_{x\in S_{\mathscr{F}}}|\widehat{q}_\alpha(x)-q_\alpha(x)|=O(h^b)+O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right).$$

4 Derivatives of conditional distribution

In this section, we focus on the estimation of the l^{th} ($l \geq 1$) order derivative $F_Y^{x^{(l)}}$ of the conditional distribution of Y given $X = x$. For this aim, we propose to construct the estimator $\widehat{F}_Y^{x^{(l)}}$ of $F_Y^{x^{(l)}}$ by

$$\widehat{F}_Y^{x^{(l)}}(y) = \frac{\sum_{i,j=1}^{n} W_{ij}(x) H^{(l)}(g^{-1}(y - Y_j))}{g^l \sum_{i,j=1}^{n} W_{ij}(x)}, \quad (3)$$

with the convention $0/0 = 0$ and W_{ij} are defined in (1).
In our previous assumptions, instead of the assumption (U2), we need (for some $l \geq 1$) the following one

(U2') For some strictly positive constants C, b_1 and b_2 and for all (x_1, x_2) in $S_{\mathscr{F}} \times B(x_1, h)$ and (y_1, y_2) in $S_{\mathbb{R}} \times B(y_1, g)$:

$$|F_Y^{x_1^{(l)}}(y_1) - F_Y^{x_2^{(l)}}(y_2)| \leq C \left(|\delta(x_1, x_2)|^{b_1} + |y_1 - y_2|^{b_2} \right).$$

(U10) H is l-times continuously differentiable.
(U11) The hypothesis (U7) is satisfied and $\lim \left(\ln n / g^{2l-1} n \Phi(h) \right) = 0$.

Then, the next result concerns the asymptotic behavior of the local linear functional estimator $F_Y^{x^{(l)}}$.

Theorem 3 *Under assumptions (U1),(U2'),(U3)–(U6) and (U10)-(U11) we have*

$$\sup_{x \in S_{\mathscr{F}}} \sup_{y \in S_{\mathbb{R}}} |\widehat{F}_Y^{x^{(l)}}(y) - F_Y^{x^{(l)}}(y)| = O(h^{b_1}) + O(g^{b_2}) + O_{a.co.} \left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{g^{2l-1} n \Phi(h)}} \right).$$

Proof. The proof of this theorem is based on the similar ideas to those used in theorem 2. The detailed demonstration can be obtained on request.

Remark 1 • As the conditional density is defined by $f_Y^x(y) = F_Y^{x^{(1)}}(y)$, so a natural estimator of $f_Y^x(y)$ is $\widehat{f}_Y^x(y) = \widehat{F}_Y^{x^{(1)}}(y)$, where $\widehat{F}_Y^{x^{(1)}}(y)$ is given by equation (3), for $l = 1$. We can deduce their pointwise and uniform almost-complete convergence under some conditions. Thus, the convergence of conditional mode.

• As the conditional hazard function is defined by $h_Y^x(y) = \frac{F_Y^{x^{(1)}}(y)}{F_Y^x(y)}$, so a natural estimator of $h_Y^x(y)$ is given by $\widehat{h}_Y^x(y) = \frac{\widehat{F}_Y^{x^{(1)}}(y)}{\widehat{F}_Y^x(y)}$. So, we can deduce their pointwise and uniform almost-complete convergence under some conditions.

5 Numerical results

In this section, we conduct an example of simulation to illustrate the performance of the local linear estimator of the conditional median (LLM). More precisely, we compare it to the conditional median kernel estimator (KM) studied in [7].
For the computation of the (LLM) and the (KM) estimators, we use the quadratic kernel $K(x) = \frac{3}{2}(1-x^2)1_{[0,1]}(x)$ and the distribution function $H(x) = \int_{-\infty}^{x} \frac{3}{4}(1-z^2)1_{[-1,1]}(z)dz$.
The bandwiths h and g are chosen by the 2-fold cross-validation method, the semimetric d is based on the derivative described in [7](see routines ”semimetric.deriv” in the website http://www.lsp.ups-tlse.fr/staph/npfda) and we take $\beta = d$ (for the LLM estimator).
Now, let us consider the following nonparametric regression model

$$Y = m(X) + \varepsilon,$$

where

$$m(X) = \left(\int_0^{\pi/3} X'(t)dt \right)^2 \quad and \quad \varepsilon \rightsquigarrow \mathcal{N}(0, 0.075).$$

The functional covariate $X(t)$ is defined, for $t \in [0, \pi/3]$ by

$$X(t) = 1 - sin(\eta_i t),$$

where $\eta_i = \frac{1}{3}\eta_{i-1} + \xi_i$, $\xi_i \rightsquigarrow \mathcal{N}(0,1))$ and are independent from η_i, which is generated independently by $\eta_0 \rightsquigarrow \mathcal{N}(0,1)$ (see Figure 1 for a sample of these curves).
Given $X = x$, we can easily see that $Y \rightsquigarrow \mathcal{N}(m(x), 0.075)$ and therefore, the conditional median functions will coincide and will be equal to $m(x)$.
In this simulation, to illustrate the performance of our estimator, we proceed as follows:
• Step 1. we split the 100-sample into a learning sample by $(X_i, Y_i)_{1 \le i \le 50}$ used to build the estimators and a testing sample by $(X_i, Y_i)_{51 \le i \le 100}$ used to make a comparison.
• Step 2. We calculate the two estimators by using the learning sample and we find the LLM ($\widehat{q}_{1/2}$) and the KM ($\widehat{q}_{1/2,KM}$) estimators of the conditional median.
• Step 3. We plot the true values ($m(X_i)$) for all i ($51 \le i \le 100$) against the predicted ones by means of the two estimators (one in each graph), this is displayed in Figure 1.
• Step 4. To be more precise, we evaluate the prediction errors given by

$$MSE(LLM) := \frac{1}{50} \sum_{j=51}^{100} \left(\widehat{q}_{1/2}(X_j) - m(X_j) \right)^2$$

and

$$MSE(KM) := \frac{1}{50} \sum_{j=51}^{100} \left(\widehat{q}_{1/2,KM}(X_j) - m(X_j) \right)^2$$

and the mean absolute errors (MAE) defined by

$$MAE(LLM) := \frac{1}{50} \sum_{j=51}^{100} |\widehat{q}_{1/2}(X_j) - m(X_j)|$$

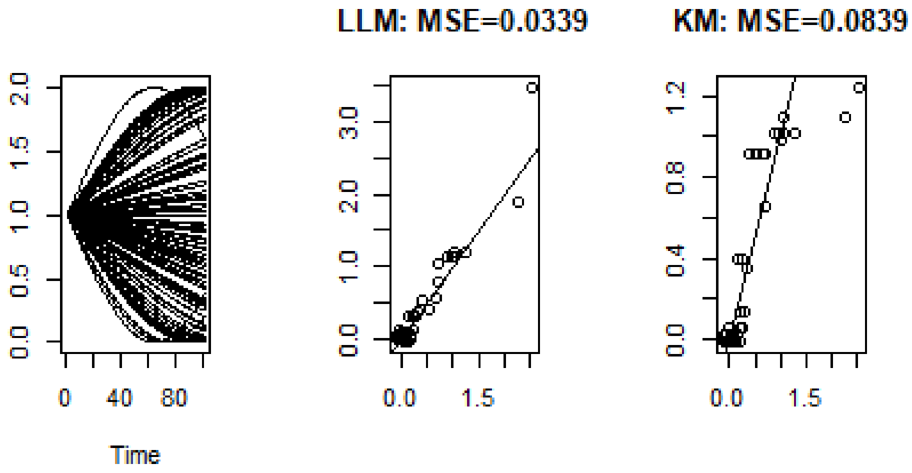

Fig. 1 From left to right the curves X_i, the LLM and the KM estimators.

and

$$MAE(KM) := \frac{1}{50}\sum_{j=51}^{100} |\widehat{q}_{1/2,KM}(X_j) - m(X_j)|.$$

The obtained results are in the Table 1.

Table 1 MSE and MAE comparaison for KM and LLM methods.

	MSE	MAE
LLM	0.0339	0.0838
KM	0.1264	0.1632

Based on these results, we see that the LLM estimator which has been introduced and studied, for independent data, in [14] also provides an acceptable performance for dependent observations.

6 Appendix

In what follows, let C be some strictly positive generic constant and for any $x \in \mathscr{F}$, and for all $i = 1, \ldots, n$:

$$K_i(x) := K(h^{-1}|\delta(X_i,x)|),\ \ \beta_i(x) := \beta(X_i,x)\ \ and\ \ H_i(y) := H(g^{-1}(y - Y_i)).$$

1. To treat the pointwise almost-complete convergence of $\widehat{F}_Y^x$, we need the following preliminary technical lemma which has the simular proof of Lemma A.1 in [11].

Lemma 7 *Under assumptions (H1), (H3) ,(H4), (H5b) and (H7), we obtain*
i) $\forall (p,l) \in \mathbb{N}^\star \times \mathbb{N}$, $E\left(K_1^p(x)|\beta_1^l(x)|\right) \leq Ch^l \Phi_x(h)$.
ii) $\forall (p_1,p_2,l_1,l_2) \in \mathbb{N}^\star \times \mathbb{N}^\star \times \mathbb{N} \times \mathbb{N}$, $E\left(K_1^{p_1}(x)K_2^{p_2}(x)|\beta_1^{l_1}(x)||\beta_2^{l_2}(x)|\right) \leq Ch^{(l_1+l_2)}(\Phi_x(h))^{a/(a-1)}$.
iii) $E\left(K_1(x)K_2(x)\beta_1^2(x)\right) > Ch^2(\Phi_x(h))^{a/(a-1)}$ *for n sufficiently large.*

As the dependence assumption reveals covariance terms, let us define for $p \in \{2,3,4\}$ and $l \in \{0,2\}$

$$S_{n,k,l}^2(x,y) = \sum_{i=1}^n \sum_{j=1}^n |Cov(\Gamma_i^{(k,l)}(x,y), \Gamma_j^{(k,l)}(x,y))|,$$

where, for $i = 1,\dots,n$

$$\Gamma_i^{(k,l)}(x,y) = \frac{1}{h^k}\left\{K_i(x)\beta_i^k(x)H_i^l(y) - E(K_i(x)\beta_i^k(x)H_i^l(y))\right\}. \quad (4)$$

Following the same lines as for proving relation (6.9) in laksaci 2011, along with the application of hypothesis (H6) and Lemma 7-i) and ii), we get

$$S_{n,k,l}^2(x,y) = O(n\Phi_x(h)). \quad (5)$$

Proof of lemma 1 The bias terms is not affected by the dependence condition. So, the proof works exactly as that of Lemma 4.3 in [14].
Proof of lemma 2 We will proceed by two steps as follows

- We can write

$$\begin{aligned}\widehat{F}_N^x(y) &= \frac{1}{n(n-1)E\left(W_{12}(x)\right)} \sum_{i,j=1}^n W_{ij}(x)H_j(y) \\ &= Q(x)[M_{2,1}^x(y)M_{4,0}^x(y) - M_{3,1}^x(y)M_{3,0}^x(y)],\end{aligned}$$

where

$$Q(x) = \frac{n^2h^2\Phi_x^2(h)}{n(n-1)E\left(W_{12}(x)\right)} \quad \textit{and} \quad M_{p,l}^x(y) = \frac{1}{n}\sum_{j=1}^n \frac{K_j(x)\beta_i^{p-2}(x)H_j^l(y)}{\Phi_x(h)}.$$

So, one has

$$\begin{aligned}\widehat{F}_N^x(y) - E\widehat{F}_N^x(y) &= Q(x)[M_{2,1}^x(y)M_{4,0}^x(y) - E(M_{2,1}^x(y)M_{4,0}^x(y)) \\ &- \ (M_{3,1}^x(y)M_{3,0}^x(y) - E(M_{3,1}^x(y)M_{3,0}^x(y)))]. \quad (6)\end{aligned}$$

We have $Q(x) = O(1)$ (see relation (14) the proof of Lemma 2 in [11]), and $E[M_{p,l}^x(y)] = O(1), \forall p,l$ (see [14] in relation (14)).
So, we have to show that for any $i = 1,2,3,4$

$$\sum_{y \in S_{\mathbb{R}}} P\left(|M_{p,l}^x(y) - E(M_{p,l}^x(y))| > \varepsilon\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right) < \infty,$$

and that almost-surely

$$cov(M^x_{2,1}(y), M^x_{4,0}(y)) = O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right) \text{ and } cov(M^x_{3,1}(y), M^x_{3,0}(y)) = O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

Firstly

$$\begin{aligned} M^x_{p,l}(y) - E(M^x_{p,l}(y)) &= \frac{1}{n\Phi_x(h)}\sum_{j=1}^{n}\left[K_j(x)\beta_i^{p-2}(x)H^l_j(y) - E(K_j(x)\beta_i^{p-2}(x)H^l_j(y))\right] \\ &:= \frac{1}{n\Phi_x(h)}\sum_{j=1}^{n}\Gamma_j^{(p-2,l)}(x,y), \end{aligned}$$

where $\Gamma_i^{(p-2,l)}(x,y)$ is defined in (4) for $k = p-2$. Now, we apply the Proposition A.11-(ii) in [7], we get, for $\varepsilon > 0$, $r \geq 1$ and for some $0 < C < \infty$

$$\begin{aligned} P\left(|M^x_{p,l}(y) - E\left[M^x_{p,l}(y)\right]| > \varepsilon\right) &= P\left(|\sum_{i=1}^{n}\Gamma_i^{(p-2,l)}(x,y)| > n\varepsilon\Phi_x(h)\right) \\ &\leq C\left[A_1(x) + A_2(x)\right], \end{aligned} \quad (7)$$

where

$$A_1(x) = \left(1 + \frac{\varepsilon^2 n^2(\Phi_x(h))^2}{rS^2_{n,p,l}(x,y)}\right)^{-r/2} \text{ and } A_2(x) = nr^{-1}\left(\frac{r}{\varepsilon n\Phi_x(h)}\right)^{(a+1)}. \quad (8)$$

Taking, for $\eta > 0$, $\varepsilon = \eta\sqrt{\frac{\ln n}{n\Phi_x(h)}}$ and $r = (\ln n)^2$ in (8) we get, by the relation (5), the condition (H8) and a Taylor expansion of $\ln(x+1)$, we get

$$A_2(x) = O(n^{-1-\nu}) \text{ and } A_2(x) = O(n^{-1-\nu'}), \quad (9)$$

for some $\nu > 0$ and $\nu' > 0$. Hence, by combining relations (7) and (9), we have

$$M^x_{p,l}(y) - E[M^x_{p,l}(y)] = O_{a.co.}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right). \quad (10)$$

Finally, by following similar arguments used to prove (5), we obtain

$$cov(M^x_{p,l}(y), M^x_{p,l}(y)) = O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

• From this last result, it is easy to obtain the uniformity on the compact $S_{\mathbb{R}}$. We omit the details because they are well known, we can see for instance the second part of the proof of Lemma 2.4 in [14].

To treat the uniform convergence of $\widehat{F}^x_Y(y)$, we need the following lemma which is the uniform version of lemma 7,

Lemma 8 *Under assumptions (U1),(U3),(U4), (U5b) and (U7), we obtain*
i) $\forall(p,l) \in \mathbb{N}^\star \times \mathbb{N}$, $\sup_{x\in S_{\mathscr{F}}} E\left(K_1^p(x)|\beta_1^l(x)|\right) \leq Ch^l\Phi(h)$.
ii) $\forall(p_1,p_2,l_1,l_2) \in \mathbb{N}^\star \times \mathbb{N}^\star \times \mathbb{N} \times \mathbb{N}$, $\sup_{x\in S_{\mathscr{F}}} E\left(K_1^{p_1}(x)K_2^{p_2}(x)|\beta_1^{l_1}(x)||\beta_2^{l_2}(x)|\right) \leq Ch^{(l_1+l_2)}(\Phi(h))^{a/(a-1)}$.
iii) $\exists n_0 \in \mathbb{N}, \forall n > n_0, \inf_{x\in S_{\mathscr{F}}} E\left(K_1(x)K_2(x)\beta_1^2(x)\right) > Ch^2(\Phi(h))^{a/(a-1)}$.

Proof of lemma 5 Following the same steps as in the proof of Lemma 2, but using Lemma 8 instead of Lemma 7, we obtain under assumptions (U1) and (U3)-(U7), for $p \in \{2,3,4\}$ and $l \in \{0,1\}$

$$\sup_{x\in S_{\mathscr{F}}} Q(x) = O(1), \quad \sup_{x\in S_{\mathscr{F}}} EM_{p,l}(x) = O(1) \tag{11}$$

and

$$\sup_{x\in S_{\mathscr{F}}} Cov\left[M_{2,1}^x(y), M_{4,0}^x(y)\right] = O\left(\frac{1}{n\Phi(h)}\right), \ \sup_{x\in S_{\mathscr{F}}} Cov\left[M_{3,1}^x(y), M_{3,0}^x(y)\right] = O\left(\frac{1}{n\Phi(h)}\right).$$

In view of (U8), this last rate is negligible with respect to $O_{ac.o}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right)$.

• Firstly, we treat the terms

$$\sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|M_{p,l}^x(y) - E(M_{p,l}^x(y))\right| \text{ for } p = 2,3,4 \textit{ and } l = 0,1.$$

For this purpose, notice that the compact set $S_{\mathbb{R}}$ can be covered by s_n open intervals centered at $\{t_j;\ 1 \leq j \leq s_n\}$, with radius $l_n = n^{-\xi-1/2}$ and $s_n = O(l_n^{-1})$ and let $t_y = \arg\min_{j\in\{1,\ldots,s_n\}} |t-t_j|$. In order to show the uniform convergence on $x \in S_{\mathscr{F}}$, we define $j(x) = \arg\min_{j\in\{1,2,\ldots,N_{r_n}(S_{\mathscr{F}})\}} d(x,x_j)$. Now, we consider the following decomposition

$$\begin{aligned}
\sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|M_{p,l}^x(y) - E(M_{p,l}^x(y))\right| \leq & \sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|M_{p,l}^x(y) - M_{p,l}^{x_{j(x)}}(y)\right| \\
& + \sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|M_{p,l}^{x_{j(x)}}(y) - M_{p,l}^{x_{j(x)}}(t_y)\right| \\
& + \sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|M_{p,l}^{x_{j(x)}}(t_y) - E(M_{p,l}^{x_{j(x)}}(t_y))\right| \\
& + \sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|E(M_{p,l}^{x_{j(x)}}(t_y)) - E(M_{p,l}^{x_{j(x)}}(y))\right| \\
& + \sup_{x\in S_{\mathscr{F}}} \sup_{y\in S_{\mathbb{R}}} \left|E(M_{p,l}^{x_{j(x)}}(y)) - E(M_{p,l}^x(y))\right| \\
:= & \sum_{k=1}^{5} T_k^{p,l}.
\end{aligned}$$

– Because of the boundless of H, the study of the term $T_1^{p,l}$ is exactly the same as that of $F_1^{p,l}$ which is defined in the proof of Lemma 5 in [11]. So we obtain

$$T_1^{p,l} = O_{ac.o}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right),$$

which entails that

$$T_5^{p,l} = O\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right).$$

– Moreover, we have

$$T_2^{p,l} \leq C\frac{l_n}{g}\sup_{x\in S_{\mathscr{F}}} M_{p,0}^{x_{j(x)}}(y)$$

In view of relations (10) and (11), the fact that $l_n = n^{-\xi-\frac{1}{2}}$ which together with $\lim_{n\to\infty} n^{\xi}g = \infty$, we can derive

$$T_2^{p,l} = O_{ac.o}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right) \quad and \quad T_4^{p,l} = O\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right).$$

– Finally, for the term $T_3^{p,l}$, by using again Corollary A.11 in [7], applying the assumption (U7) and the fact that $s_n = O(l_n) = O(\eta^{\xi+\frac{1}{2}})$, we get

$$P\left(T_3^{p,l} > \eta\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right)$$
$$= P\left(\sup_{x\in S_{\mathscr{F}}}\sup_{y\in S_{\mathbb{R}}}\left|M_{p,l}^{x_{j(x)}}(t_y) - E(M_{p,l}^{x_{j(x)}}(t_y))\right| > \eta\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right)$$
$$\leq N_{r_n}(S_{\mathscr{F}})s_n \times$$
$$\max_{t_y\in\{t_1,\ldots,t_{s_n}\}}\max_{x_{j(x)}\in\{x_1,\ldots,x_{N_{r_n}(S_{\mathscr{F}})}\}} P\left(\left|M_{p,l}^{x_{j(x)}}(t_y) - E(M_{p,l}^{x_{j(x)}}(t_y))\right| > \eta\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right)$$
$$< Cn^{-1-\nu''},$$

for some $\nu'' > 0$, which means that

$$T_3^{p,l} = O_{ac.o}\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\ln n}{n}\right)}{n\Phi(h)}}\right).$$

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