Enhanced Arithmetic Operations for Fuzzy Numbers: Theory and Applications

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Abstract

Fuzzy arithmetic is essential for handling uncertainty in decision-making and modeling, yet existing approaches such as the extension principle, α-cut method, and Hukuhara difference suffer from loss of information and excessive imprecision. In particular, they do not allow recovery of the original fuzzy numbers once operations are applied. To address this limitation, we propose a novel reversible arithmetic framework for triangular fuzzy numbers that ensures both consistency and retrievability of operands. The method is applied to solve fuzzy scalar and system of linear equations, including economic supply–demand models, demonstrating enhanced accuracy and interpretability compared with traditional approaches. The proposed framework provides a stronger mathematical basis for fuzzy arithmetic and broadens its applicability to real-world modeling and decision support.

keyword : Triangular fuzzy number, reversible fuzzy arithmetic, fuzzy linear equation, supply-demand modeling.

1 Introduction

Fuzzy numbers are commonly used for dealing with an uncertainty of possibilistic nature, decision-making support and modeling real life applications in more realistic form. Since the introduction of fuzzy set theory by [1], various arithmetic operations on fuzzy numbers have been developed to handle imprecise data in different practical applications. Working with fuzzy numbers for arithmetic operations like addition, subtraction, multiplication, and division can be tricky because of the imprecision that comes with them.

In the early methods of fuzzy arithmetic operation, the extension principle was used, where operations were performed by applying interval arithmetic to different α-cut levels. [2] introduced a method for performing arithmetic operations based on extension principle and α-cuts. When performing operations using this approach, the resulting fuzzy number often had a wider support than expected. Later, [3] improved this method by using different ways to represent alpha levels, but the problem of increasing fuzziness still remained.

Alternative methods such as the Hukuhara difference (H difference) were introduced to maintain the closure property of fuzzy numbers. However, as studied by [4], the H-difference is not always well-defined, particularly when dealing with asymmetric fuzzy numbers. [5] ,[6] proposed a generalization of the Hukuhara difference and division, which aimed to extend the traditional Hukuhara difference by providing a more flexible approach to interval and fuzzy arithmetic. However, this method still faced challenges in practical applications, particularly due to increased computational complexity which arises from interval analysis, set operations, and optimization techniques. Moreover, it does not handle non-convex, or asymmetric fuzzy numbers easily for practical purposes.

The α-cut method yields more reliable results according to [7] but takes more time to compute, compared to the method without α-cuts which provides faster but less precise results. The authors emphasized finding a balance between precise calculations and efficient execution in fuzzy arithmetic procedures. [8] studied arithmetic operations on generalized intuitionistic fuzzy numbers and applied them to transportation problems, demonstrating the practical implications of fuzzy arithmetic. [9] investigated solving interval fuzzy linear programming problems using the alpha-cut method, further emphasizing the importance of structured approaches in fuzzy arithmetic.

Further contributions to fuzzy arithmetic, [10] introduced a computational method for performing fuzzy arithmetic operations on triangular fuzzy numbers using the extension principle. While this method provided a structured way to perform arithmetic operations, it was complicated and not easily applicable to non-triangular fuzzy sets. Additionally, [11] introduced new arithmetic operations for non-normal fuzzy sets using compatibility concepts, aiming to improve the consistency of fuzzy arithmetic. [12] proposed a class of fuzzy numbers induced by probability density functions and explored their arithmetic operations, expanding the theoretical foundation of fuzzy arithmetic.

Even though many improvements have been made, fuzzy arithmetic operations methods still have problems. They often make the results too imprecise, do not work well for certain types of fuzzy numbers and also we can not get original fuzzy operands after applying the method. Because of these issues, there is a need for a better method that gives more accurate results, is easier to compute.

This paper introduces a novel approach to fuzzy arithmetic operations that addresses existing limitations and makes them more practical for real-world applications. The proposed method overcomes a significant limitation by allowing the original fuzzy numbers to be recovered from the results. Our method preserves consistency in fuzzy arithmetic i.e. if $\tilde{A} * \tilde{B} = \tilde{C}$, where $*$ represents fuzzy arithmetic operations, then we can also compute $\tilde{A}$ and $\tilde{B}$ from resultant fuzzy number. This property improves the reliability of our method in real world problems.

Using the proposed novel approach, we provide illustrative examples for arithmetic operations by considering different cases according to different lengths and types of triangular fuzzy numbers viz. positive, negative, and including zero. Also, we solve fuzzy linear equations of the form $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \otimes \tilde{X} = \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are triangular fuzzy numbers. In the case of equation $\tilde{A} \otimes \tilde{X} = \tilde{B}$, we consider only fully positive, or fully negative fuzzy numbers. Furthermore, we apply the proposed method to solve fuzzy linear equations and demonstrate its effectiveness in mathematical modeling problems, including supply–demand models and fuzzy system analysis.

The structure of this paper is organized as follows, in section 2, we provide the notion of fuzzy numbers along with relevant concepts. In section 3, we present our novel method for arithmetic operations on fuzzy number and how the original numbers can be recovered after applying the operations and show that the results are consistent. In Section 4, we use numerical examples to show how our method works with different types of triangular fuzzy numbers and we show that the proposed technique is suitable for multiple cases. Finally, Section 5 applies the proposed approach to solving systems of fuzzy linear equations, including illustrative examples from mathematical modeling such as supply–demand problem.

2 Preliminaries

Definition 2.1 *A fuzzy set $\tilde{A}$ is represented as a collection of ordered pairs:*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in \tilde{A}, \mu_{\tilde{A}}(x) \in [0,1]\}$$

where,

- *x is an element from a classical set $\tilde{A}$,*
- *$\mu_{\tilde{A}}(x)$ is the membership function, which assigns a degree of membership to each element x within the interval [0,1].*

The membership function $\mu_{\tilde{A}}(x)$ represents the degree to which the element x belongs to the fuzzy set $\tilde{A}$. A value of 0 means x is not in the fuzzy set, while a value of 1 means it fully belongs to the fuzzy set. Intermediate values indicate partial membership.

Definition 2.2 *An α-cut (or alpha cut) of a fuzzy set or fuzzy number $\tilde{A}$ is a crisp set that contains all elements of the fuzzy set with a membership degree greater than or equal to a specified value of α, where, $\alpha \in [0,1]$.*

Mathematically, the α-cut of a fuzzy set $\tilde{A}$ is denoted and given as:

$${}^{\alpha}\tilde{A} = \{x \in \tilde{A} : \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Definition 2.3 *The support of a fuzzy set $\tilde{A}$ within a universal set X is defined as a crisp set of all elements in X that have a non-zero membership value in $\tilde{A}$.*

Mathematically, it is represented as:

$$S(\tilde{A}) = \{x \in X : \mu_{\tilde{A}}(x) > 0\}$$

Definition 2.4 *Let R be the set of real numbers and $\tilde{A} : R \to [0,1]$ be a fuzzy set. We say that $\tilde{A}$ is a fuzzy number if it satisfies at least the following three properties:*

1. *$\tilde{A}$ is normal, i.e., there exists $x_0 \in R$ such that $\tilde{A}(x_0) = 1$,*
2. *$\tilde{A}^{\alpha}$ must be a closed interval for every $\alpha \in (0,1]$ i.e. $\tilde{A}$ must be a convex fuzzy set, that is, $\tilde{A}(\lambda x + (1-\lambda)y) \geq min(\tilde{A}(x), \tilde{A}(y))$, whenever $x, y \in R$ and $\lambda \in [0,1]$,*
3. *the support of $\tilde{A}$, ${}^{0+}\tilde{A}$ must be bounded. i.e. the membership function of $\tilde{A}$ must be piecewise continuous.*

Definition 2.5 *An arbitrary fuzzy number $\tilde{A}$ in parametric form is a pair $\left[\underline{A}(r), \overline{A}(r)\right]$ of functions $\underline{A}(r)$ and $\overline{A}(r)$ for any $r \in [0,1]$, which satisfies the following properties:*

1. *$\underline{A}(r)$ is a bounded monotonic increasing left continuous function in $(0,1]$,*
2. *$\overline{A}(r)$ is a bounded monotonic decreasing left continuous function in $(0,1]$,*
3. *$\underline{A}(r) \leq \overline{A}(r)$, $0 \leq r \leq 1$.*

Definition 2.6 *A fuzzy number $\tilde{A}$ is called a triangular fuzzy number (TFN) if it can be represented by a triplet of real numbers (a_1, a_2, a_3), where $a_1 \leq a_2 \leq a_3$.*

The membership function of a triangular fuzzy number is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{for } a_1 < x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & \text{for } a_2 < x \leq a_3 \\ 0, & otherwise \end{cases}$$

This membership function rises linearly from 0 to 1 as x moves from a_1 to a_2 and then decreases linearly from 1 back to 0 as x moves from a_2 to a_3.
Any set containing only a triangular fuzzy number is called a triangular fuzzy set.
The α-cut of a triangular fuzzy number with respect to a level α is given by the interval:

$$^{\alpha}\tilde{A} = [\underline{A}(\alpha), \overline{A}(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]$$

Also, length($\tilde{A}$) means the length of interval of $^{0}\tilde{A}$ which is $a_3 - a_1$.

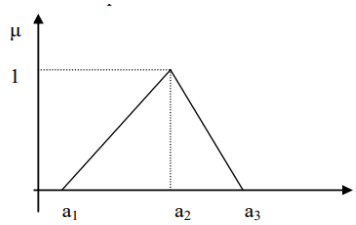

Figure 1: Triangular Fuzzy Number $\tilde{A}$= (a_1, a_2, a_3)

Definition 2.7 *The interval-based fuzzy arithmetic operations for fuzzy numbers $\tilde{x}$ and $\tilde{y}$, corresponding $\alpha - cut$ are $^{\alpha}\tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)]$ and $^{\alpha}\tilde{y} = [\underline{y}(\alpha), \overline{y}(\alpha)]$ can be described as follows:*

1. **Equality of two fuzzy number :**

$$\tilde{x} = \tilde{y} \text{ iff } \underline{x}(\alpha) = \underline{y}(\alpha) \text{ and } \overline{x}(\alpha) = \overline{y}(\alpha)$$

2. **Addition of two fuzzy number :**

$$^{\alpha}(\tilde{x} \oplus \tilde{y}) = {}^{\alpha}(\tilde{x}) \oplus {}^{\alpha}(\tilde{y}) = [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)]$$

3. **Multiplication of two fuzzy number :**

$$^{\alpha}(\tilde{x} \otimes \tilde{y}) =^{\alpha} (\tilde{x}) \otimes^{\alpha} (\tilde{y}) = [min(\underline{x}(\alpha) \times \underline{y}(\alpha), \underline{x}(\alpha) \times \overline{y}(\alpha), \overline{x}(\alpha) \times \underline{y}(\alpha), \overline{x}(\alpha) \times \overline{y}(\alpha)),$$
$$max(\underline{x}(\alpha) \times \underline{y}(\alpha), \underline{x}(\alpha) \times \overline{y}(\alpha), \overline{x}(\alpha) \times \underline{y}(\alpha), \overline{x}(\alpha) \times \overline{y}(\alpha))]$$

4. **Division of two fuzzy number :**

$$^{\alpha}(\tilde{x}/\tilde{y}) =^{\alpha} (\tilde{x})/^{\alpha}(\tilde{y}) = [min(\underline{x}(\alpha)/\underline{y}(\alpha), \underline{x}(\alpha)/\overline{y}(\alpha), \overline{x}(\alpha)/\underline{y}(\alpha), \overline{x}(\alpha)/\overline{y}(\alpha)),$$
$$max(\underline{x}(\alpha)/\underline{y}(\alpha), \underline{x}(\alpha)/\overline{y}(\alpha), \overline{x}(\alpha)/\underline{y}(\alpha), \overline{x}(\alpha)/\overline{y}(\alpha))],$$

5. **Scalar Multiplication** (for scalar k):

   - if $k \geq 0$

$$k \otimes^{\alpha} \tilde{x} = [k \otimes \underline{x}(\alpha), k \otimes \overline{x}(\alpha)]$$

   - if $k < 0$

$$k \otimes^{\alpha} \tilde{x} = [k \otimes \overline{x}(\alpha), k \otimes \underline{x}(\alpha)]$$

Definition 2.8 ***First Decomposition Theorem***
For every $A \in \mathcal{F}(X)$

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A$$

where, ${}_{\alpha}A$ is a special fuzzy set and defined as ${}_{\alpha}A = \alpha \cdot^{\alpha} A$ and $\bigcup$ denotes the standard fuzzy union.

Definition 2.9 *A positive triangular fuzzy number $\tilde{A}$ is represented as $\tilde{A} = (a_1, a_2, a_3)$, where all the values a_1, a_2 and a_3 are greater than zero.*
i.e. $a_i > 0$ for $i = 1, 2, 3$.

Definition 2.10 *A negative triangular fuzzy number $\tilde{A}$ is represented as $\tilde{A} = (a_1, a_2, a_3)$, where all the values a_1, a_2 and a_3 are less than zero.*
i.e. $a_i < 0$ for $i = 1, 2, 3$.

Note 1 *A negative triangular fuzzy number can be obtained by multiplying a positive triangular fuzzy number by (-1).*

For example, if we have a positive triangular fuzzy number $\tilde{A} = (5, 6, 7)$, then its negative can be written as

$$\tilde{B} = -\tilde{A} = -1 * (5, 6, 7) = (-7, -6, -5).$$

Definition 2.11 *A triangular fuzzy number, denoted as $\tilde{A} = (a_1, a_2, a_3)$, is classified as an around-zero triangular fuzzy number if $a_1 < 0$ and $a_3 > 0$.*

For example : The triangular fuzzy number $\tilde{A} = (-5, -3, 1)$ is considered as an around zero triangular fuzzy number.

Henceforth, we consider triangular fuzzy numbers.

3 Arithmetic operation on triangular fuzzy number

Interval based arithmetic operations on fuzzy numbers, given in previous section, have certain limitation. For example, when $\tilde{C} = \tilde{A} \circledast \tilde{B}$, retrieving $\tilde{A}$ and $\tilde{B}$ from resulting $\tilde{C}$ using definition (2.7) is not possible.
We propose a new definition for these operations, which eliminates this limitation. The definition and the technique to perform the arithmetic operation on two fuzzy numbers are described in the following:

Let $\tilde{A} = (a_1, a_2, a_3)$, $\tilde{B} = (b_1, b_2, b_3)$ and $\tilde{C} = (c_1, c_2, c_3)$ be triangular fuzzy numbers. Let the α-cuts of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ be given as

$${}^{\alpha}\tilde{A} = [a_l, a_u] \text{ and } {}^{\alpha}\tilde{B} = [b_l, b_u].$$

To define the operations $\tilde{A} \circledast \tilde{B}$, where $\circledast$ represents $\{\oplus, \ominus, \otimes, \oslash\}$ corresponding to addition, subtraction, multiplication and division of fuzzy numbers, we compute the set L given as:

$$L = [a_l * b_l, a_l * b_u, a_u * b_l, a_u * b_u],$$

where, $*$ can be $\{+,-,\times,/\}$ and denotes the usual arithmetic operations on real numbers.

Next, we apply the function $fsort()$ on L. This arranges the four elements of L for $\forall\ \alpha \in [0,1]$ in the increasing order. We denote the ordered elements as r_1, r_2, r_3, r_4, such that $r_1 < r_2 < r_3 < r_4$.
Thus,

$$R = fsort() = [r_1, r_2, r_3, r_4].$$

Using these elements of R, we define the result of arithmetic operations on fuzzy numbers as follows:

Definition 3.1 *The arithmetic operations on fuzzy numbers $\tilde{A}$ and $\tilde{B}$ denoted by $\tilde{C}$ is defined as*

$$\tilde{C} = \tilde{A} \circledast \tilde{B}.$$

Applying $\alpha-$cuts on both sides, we get:

$$^{\alpha}\tilde{C} =^{\alpha} \tilde{A} *^{\alpha} \tilde{B}.$$

Since $^{\alpha}\tilde{A} *^{\alpha} \tilde{B}$ is computed by the steps above, we obtain

$$^{\alpha}\tilde{C} = [r_1, r_4]. \tag{1}$$

where, r_1 and r_4 are the first and last elements of R defined above.
Then, by Theorem (2.8), putting $\alpha = 1$ gives core (denoted by c_2), and $\alpha = 0$ gives support, given as $[c_1, c_3]$.
Hence, the triangular fuzzy number is given by:

$$\tilde{C} = (c_1, c_2, c_3).$$

Example: Let $\tilde{A} = (-7,-4,-2)$ and $\tilde{B} = (-6,-5,-4)$. The $\alpha-$cut of A is $^{\alpha}\tilde{A} = [3\alpha-7, -2-2\alpha]$ and $\alpha-$cut of B is $^{\alpha}\tilde{B} = [\alpha-6, -4-\alpha]$.
Addition of $\tilde{A}$ and $\tilde{B}$ be $\tilde{C}$. Then to compute $^{\alpha}\tilde{C}$, Obtain L as,

$$L = [3\alpha - 7 + (\alpha - 6), 3\alpha - 7 + (-4 - \alpha), -2 - 2\alpha + (\alpha - 6), -2 - 2\alpha + (-4 - \alpha)]$$
$$L = [4\alpha - 13, 2\alpha - 11, -8 - \alpha, -6 - 3\alpha].$$

Applying the $fsort()$ function to L as,

$$fsort(L) = [4\alpha - 13, 2\alpha - 11, -8 - \alpha, -6 - 3\alpha].$$

Therefore, $^{\alpha}\tilde{C} = [4\alpha - 13, -6 - 3\alpha]$. Using Theorem (2.8), $\tilde{C} = (-13, -9, -6)$.

The advantage of this new definition for the four basic fuzzy arithmetic operations $(\oplus, \ominus, \otimes, \oslash)$ is that the resulting fuzzy number $\tilde{C}$ allows us to retrieve the original fuzzy numbers $\tilde{A}$ and $\tilde{B}$,

- If $\tilde{A} \oplus \tilde{B} = \tilde{C}$, then $\tilde{A} = \tilde{C} \ominus \tilde{B}$, $\tilde{B} = \tilde{C} \ominus \tilde{A}$.
- If $\tilde{A} \ominus \tilde{B} = \tilde{C}$, then $\tilde{A} = \tilde{C} \oplus \tilde{B}$, $\tilde{B} = \tilde{A} \ominus \tilde{C}$.
- If $\tilde{A} \otimes \tilde{B} = \tilde{C}$, then $\tilde{A} = \tilde{C} \oslash \tilde{B}$, $\tilde{B} = \tilde{C} \oslash \tilde{A}$.
- If $\tilde{A} \oslash \tilde{B} = \tilde{C}$, then $\tilde{A} = \tilde{C} \otimes \tilde{B}$, $\tilde{B} = \tilde{A} \oslash \tilde{C}$.

However in these operations, in the last step after applying $fsort()$ function, we have to consider the elements $[r_2, r_3]$ for getting the original operands ${}^{\alpha}\tilde{A}$ and ${}^{\alpha}\tilde{B}$.
Note:

1. For multiplication and division operations, fuzzy numbers $\tilde{A}$ and $\tilde{B}$ must be either strictly positive or strictly negative.
2. This definition is easily applicable to trapezoidal fuzzy numbers. However, for gaussian fuzzy number, one needs to check.

The newly proposed arithmetic operations satisfies various properties as shown in the next section.

4 Properties of Arithmetic Operations on Fuzzy Numbers

We now establish the fundamental properties of the fuzzy arithmetic operations $\oplus, \ominus, \otimes, \oslash$ on triangular fuzzy numbers, defined via α-cuts and the $fsort()$ function.

Theorem 4.1 (Closure) *If $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers, then $\tilde{A} \circledast \tilde{B}$ is also a fuzzy number, where $\circledast \in \{\oplus, \ominus, \otimes, \oslash\}$ represents an operation on fuzzy numbers and $* \in \{+, -, \times, /\}$ represents an equivalent operations on real numbers.*

Proof : For each $\alpha \in [0, 1]$, the α-cuts of $\tilde{A}$ and $\tilde{B}$ are closed bounded intervals ${}^{\alpha}\tilde{A} = [A_l(\alpha), A_u(\alpha)]$ and ${}^{\alpha}\tilde{B} = [B_l(\alpha), B_u(\alpha)]$. Applying interval arithmetic gives four elements $L = [A_l * B_l, A_l * B_u, A_u * B_l, A_u * B_u]$. After applying $fsort()$, we obtain $R = [r_1, r_2, r_3, r_4]$, and hence ${}^{\alpha}\tilde{C} = [r_1, r_4]$. This is a valid interval for every α, so $\tilde{C}$ is also a fuzzy number.

Theorem 4.2 (Commutativity) *For fuzzy numbers $\tilde{A}, \tilde{B}$,*

$$\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A}, \quad \tilde{A} \otimes \tilde{B} = \tilde{B} \otimes \tilde{A}.$$

Proof : For each α, we compute

$$L = [A_l * B_l, A_l * B_u, A_u * B_l, A_u * B_u] = [B_l * A_l, B_l * A_u, B_u * A_l, B_u * A_u].$$

Exchanging $\tilde{A}$ and $\tilde{B}$ does not change L, so the $fsort()$ result R is identical. Thus ${}^{\alpha}(\tilde{A} * \tilde{B}) =^{\alpha} (\tilde{B} * \tilde{A})$ for all α, which proves that the operations are commutative.

Theorem 4.3 (Associativity) *For fuzzy numbers A, B, C,*

$$(\tilde{A} \oplus \tilde{B}) \oplus \tilde{C} = \tilde{A} \oplus (\tilde{B} \oplus \tilde{C}), \quad (\tilde{A} \otimes \tilde{B}) \otimes \tilde{C} = \tilde{A} \otimes (\tilde{B} \otimes \tilde{C}).$$

Proof : For each α, the α-cut is an interval in $\mathbb{R}$. Interval addition and multiplication are associative in $\mathbb{R}$, and since $fsort()$ only preserves ordering, the resulting intervals remain consistent. Hence associativity holds for fuzzy numbers.

Theorem 4.4 (Identity Elements) *For any fuzzy number* $\tilde{A}$*,*

$$\tilde{A} \oplus \tilde{0} = \tilde{A}, \quad \tilde{A} \otimes \tilde{1} = \tilde{A},$$

where $\tilde{0} = (a, 0, b)$ *and* $\tilde{1} = (1 - a, 1, 1 + b)$*.*

Proof: For each α, the α-cuts of the additive identity element is given by

$$^{\alpha}\tilde{0} = [a(1-\alpha),\, b(1-\alpha)].$$

For each α, we compute

$$L = [A_l+0_l,\ A_l+0_u,\ A_u+0_l,\ A_u+0_u] = [A_l+a(1-\alpha),\ A_l+b(1-\alpha),\ A_u+a(1-\alpha),\ A_u+b(1-\alpha)]$$

At core ($\alpha = 1$),

$$^{\alpha}(\tilde{A} \oplus \tilde{0}) = [A_l, A_l, A_u, A_u] = [A_l, A_u] =^{\alpha} \tilde{A}.$$

Hence, $\tilde{0}$ act as identity element for fuzzy addition.

Now, for $\tilde{A} \otimes \tilde{1} = \tilde{A}$. For each α, the α-cut of multiplicative identity element is given by

$$^{\alpha}\tilde{1} = [1 + a(\alpha - 1),\, 1 + b(1 - \alpha)].$$

For each α, we compute

$$^{\alpha}(\tilde{A} \otimes \tilde{1}) = [A_l(1 + a(\alpha - 1)),\ A_u(1 + a(\alpha - 1)), A_l(1 + b(1 - \alpha)), A_u(1 + b(1 - \alpha))].$$

At core ($\alpha = 1$),

$$^{\alpha}(\tilde{A} \otimes \tilde{1}) = [A_l, A_l, A_u, A_u] = [A_l, A_u] =^{\alpha} \tilde{A}.$$

Hence, $\tilde{1}$ act as identity element for fuzzy multiplication.

Theorem 4.5 (Cancellation Law) *If* $\tilde{A} \otimes \tilde{B} = \tilde{A} \otimes \tilde{C}$ *and* $\tilde{A} \neq \tilde{0}$*, then* $\tilde{B} = \tilde{C}$*.*

Proof : For each α, assume $^{\alpha}(\tilde{A} \otimes \tilde{B}) =^{\alpha} (\tilde{A} \otimes \tilde{C})$. That is,

$$[A_l(\alpha), A_u(\alpha)] * [B_l(\alpha), B_u(\alpha)] = [A_l(\alpha), A_u(\alpha)] * [C_l(\alpha), C_u(\alpha)].$$

Since $^{\alpha}\tilde{A}$ is a non-degenerate interval (because $\tilde{A} \neq \tilde{0}$ or $0 \notin \tilde{A}$), cancellation in real interval arithmetic implies $[B_l(\alpha), B_u(\alpha)] = [C_l(\alpha), C_u(\alpha)]$ for all α. Hence, $\tilde{B} = \tilde{C}$.

Theorem 4.6 (Zero-Product Property) *If* $\tilde{A} \otimes \tilde{B} = \tilde{0}$*, then either* $\tilde{A} = \tilde{0}$ *or* $\tilde{B} = \tilde{0}$*.*

Proof : Suppose $^{\alpha}(\tilde{A} \otimes \tilde{B}) = [0, 0]$ for all α. Then all four elements in

$$L = [A_l(\alpha) * B_l(\alpha),\ A_l(\alpha) * B_u(\alpha),\ A_u(\alpha) * B_l(\alpha),\ A_u(\alpha) * B_u(\alpha)]$$

are zero for each α. This implies that either $[A_l(\alpha), A_u(\alpha)] = [0, 0]$ or $[B_l(\alpha), B_u(\alpha)] = [0, 0]$ for all α. Therefore, $\tilde{A} = \tilde{0}$ or $\tilde{B} = \tilde{0}$.

Theorem 4.7 (Additive Inverse) *For any fuzzy number* $\tilde{A}$*,*

$$\tilde{A} \oplus (-\tilde{A}) = \tilde{0},$$

where $-\tilde{A}$ *is the additive inverse of* $\tilde{A}$*.*

Proof : For each α, the α-cut of $\tilde{A}$ is ${}^{\alpha}\tilde{A} = [A_l(\alpha), A_u(\alpha)]$. The α-cut of $-\tilde{A}$ is ${}^{\alpha}(-\tilde{A}) = [-A_u(\alpha), -A_l(\alpha)]$. Using the proposed method, we form

$$L = [A_l + (-A_u),\ A_l + (-A_l),\ A_u + (-A_u),\ A_u + (-A_l)] = [A_l - A_u,\ 0,\ 0,\ A_u - A_l].$$

Applying $fsort()$ arranges these in increasing order by taking $\alpha = 1$, giving $R = [0, 0, 0, 0]$. Thus,

$${}^{\alpha}(\tilde{A} \oplus (-\tilde{A})) = [0, 0].$$

Hence,

$$\tilde{A} \oplus (-\tilde{A}) = \tilde{0}.$$

Theorem 4.8 (Multiplicative Inverse) *For any fuzzy number $\tilde{A} \neq \tilde{0}$,*

$$\tilde{A} \otimes \tilde{A}^{-1} = \tilde{1},$$

where $\tilde{A}^{-1}$ is the multiplicative inverse of $\tilde{A}$.

Proof : For each α, the α-cut of $\tilde{A}$ is ${}^{\alpha}\tilde{A} = [A_l(\alpha), A_u(\alpha)]$, with $0 \notin [A_l(\alpha), A_u(\alpha)]$. The α-cut of $\tilde{A}^{-1}$ is

$${}^{\alpha}\tilde{A}^{-1} = \left[\frac{1}{A_u(\alpha)}, \frac{1}{A_l(\alpha)}\right].$$

Multiplying ${}^{\alpha}\tilde{A}$ and ${}^{\alpha}\tilde{A}^{-1}$ gives four elements, all equal to 1 at $\alpha = 1$. Hence after $fsort()$,

$${}^{\alpha}(\tilde{A} \otimes \tilde{A}^{-1}) = [1, 1].$$

Therefore, $\tilde{A} \otimes \tilde{A}^{-1} = \tilde{1}$.

In the following section, we provide illustrative examples of arithmetic operations by considering different cases based on the lengths and types of TFNs.

5 Illustrative Examples

Case 1: length($\tilde{A}$)> length($\tilde{B}$)

Example 1. ($\tilde{A}$ and $\tilde{B}$ both are positive TFN.)

Consider two positive fuzzy numbers with $\tilde{A} = (3, 5, 8)$ and $\tilde{B} = (4, 6, 7)$.
The α-cut of $\tilde{A}$ is ${}^{\alpha}\tilde{A} = [2\alpha + 3, 8 - 3\alpha] = [a_l, a_u]$ and α-cut of $\tilde{B}$ is ${}^{\alpha}\tilde{B} = [2\alpha + 4, 7 - \alpha] = [b_l, b_u]$, where $\alpha \in [0, 1]$.

Addition:

Let addition of $\tilde{A}$ and $\tilde{B}$ be $\tilde{C}$, i.e. $\tilde{A} \oplus \tilde{B} = \tilde{C}$.
Now, ${}^{\alpha}\tilde{C} = {}^{\alpha}\tilde{A} + {}^{\alpha}\tilde{B}$, then to compute ${}^{\alpha}\tilde{C}$, following steps as given in our definition. We have
Step-1: Obtain L as,

$$L = [2\alpha + 3 + 2\alpha + 4, 2\alpha + 3 + 7 - \alpha, 8 - 3\alpha + 2\alpha + 4, 8 - 3\alpha + 7 - \alpha]$$
$$L = [4\alpha + 7, \alpha + 10, 12 - \alpha, 15 - 4\alpha].$$

Step-2: Now apply the fsort() function, to obtain R as

$$R = fsort(L) = [4\alpha + 7, \alpha + 10, 12 - \alpha, 15 - 4\alpha].$$

Step-3: Select the first and last elements of R to obtain α-cut of $\tilde{C}$.
Therefore,

$$^{\alpha}\tilde{C} = [4\alpha + 7, 15 - 4\alpha].$$

Using FDT, $\tilde{C} = (7, 11, 15)$.
Calculation for original fuzzy number:
Now, we want to obtain the numbers that we started with i.e. $\tilde{A}$ and $\tilde{B}$ from $\tilde{C}$.
For $\tilde{A} = \tilde{C} \ominus \tilde{B}$, we consider α-cuts of $\tilde{C}$ and $\tilde{B}$ i.e. $^{\alpha}\tilde{C} \ominus {}^{\alpha}\tilde{B}$ and we apply step $1, 2$ and lastly in step-3 considering interval $[r_2, r_3]$ given us $^{\alpha}\tilde{A}$ as follows:
Step-1:

$$L = [4\alpha + 7 - (4 + 2\alpha), 4\alpha + 7 - (7 - \alpha), 15 - 4\alpha - (4 + 2\alpha), 15 - 4\alpha - (7 - \alpha)]$$
$$L = [2\alpha + 3, 5\alpha, 11 - 6\alpha, 8 - 3\alpha]$$

Step-2: Sorted L, $R = [5\alpha, 2\alpha + 3, 8 - 3\alpha, 11 - 6\alpha]$.
Step-3: From R selecting $[r_2, r_3]$ gives,

$$^{\alpha}\tilde{A} = [2\alpha + 3, 8 - 3\alpha].$$

Using FDT, $\tilde{A} = (3, 5, 8)$.
For $\tilde{B}$, the relation is $\tilde{C} \ominus \tilde{A}$. Taking α-cuts of the elements on R.H.S., i.e.$^{\alpha}\tilde{C} - {}^{\alpha}\tilde{A}$ and considering the steps of getting original numbers:
Step-1: $L = [2\alpha + 4, 7\alpha - 1, 12 - 6\alpha, 7 - \alpha]$.
Step-2: Sorted L, $R = [7\alpha - 1, 4 + 2\alpha, 7 - \alpha, 12 - 6\alpha]$.
Step-3: From R selecting $[r_2, r_3]$ gives,

$$^{\alpha}\tilde{B} = [4 + 2\alpha, 7 - \alpha].$$

Using FDT, $\tilde{B} = (4, 6, 7)$.

Subtraction:

Let subtraction from $\tilde{A}$ to $\tilde{B}$ be $\tilde{D}$, i.e $\tilde{A} \ominus \tilde{B} = \tilde{D}$.
Now, $^{\alpha}\tilde{D} = {}^{\alpha}\tilde{A} - {}^{\alpha}\tilde{B}$, the following steps are used to compute $^{\alpha}\tilde{D}$:
Step-1:

$$L = [2\alpha{+}3{-}(2\alpha{+}4), 2\alpha{+}3{-}(7{-}\alpha), 8{-}3\alpha{-}(2\alpha{+}4), 8{-}3\alpha{-}(7{-}\alpha)] = [-1, 3\alpha{-}4, -5\alpha{+}4, 1{-}2\alpha]$$

Step-2: Apply the fsort() function, to obtain R as

$$R = fsort(L) = [3\alpha - 4, -1, 1 - 2\alpha, -5\alpha + 4].$$

Step-3: Select the first and last elements of R to obtain α-cut of $\tilde{D}$.
Therefore,

$$^{\alpha}\tilde{D} = [3\alpha - 4, -5\alpha + 4].$$

Using FDT, $\tilde{D} = (-4, -1, 4)$.
Calculation for original fuzzy number:
Now we want to obtain the number $\tilde{A}$ from $\tilde{D} \ominus \tilde{B}$, taking $\alpha-$cuts, we have
Step-1: $L = [5\alpha, 2\alpha + 3, 8 - 3\alpha, 11 - 6\alpha]$

Step-2: Applying the fsort() function, R will be, $R = [5\alpha, 2\alpha + 3, 8 - 3\alpha, 11 - 6\alpha]$.
Step-3: Select the 2^{nd} and 3^{rd} elements of R, which gives the a_l, a_u, therefore

$$^{\alpha}\tilde{A} = [2\alpha + 3, 8 - 3\alpha].$$

Using FDT, $\tilde{A} = (3, 5, 8)$.
Now, We find $\tilde{B}$ from $\tilde{A} \ominus \tilde{D}$ as follows by taking α-cuts as:
Step-1: $L = [7 - \alpha, 7\alpha - 1, 12 - 6\alpha, 4 + 2\alpha]$
Step-2: $R = [7\alpha - 1, 4 + 2\alpha, 7 - \alpha, 12 - 6\alpha]$.
Step-3: Selecting $[r_2, r_3]$ from R gives,

$$^{\alpha}\tilde{B} = [4 + 2\alpha, 7 - \alpha].$$

Using FDT, $\tilde{B} = (4, 5, 7)$.

Multiplication:

Let multiplication of $\tilde{A}$ and $\tilde{B}$ be $\tilde{E}$, *i.e.*$\tilde{A} \otimes \tilde{B} = \tilde{E}$. The computation of $\tilde{E}$ involves the following steps:
Step-1: For $^{\alpha}\tilde{E} = {}^{\alpha}\tilde{A} \otimes {}^{\alpha}\tilde{B}$, obtain L as,

$$L = [(2\alpha + 3) \times (2\alpha + 4), (2\alpha + 3) \times (7 - \alpha), (8 - 3\alpha) \times (4 + 2\alpha), (8 - 3\alpha) \times (7 - \alpha)]$$
$$L = [4\alpha^2 + 14\alpha + 12, -2\alpha^2 + 11\alpha + 21, -6\alpha^2 + 4\alpha + 32, 3\alpha^2 - 29\alpha + 56]$$

Step-2: Now apply the fsort() function,

$$R = [4\alpha^2 + 14\alpha + 12, -2\alpha^2 + 11\alpha + 21, -6\alpha^2 + 4\alpha + 32, 3\alpha^2 - 29\alpha + 56].$$

Step-3: Select the first and last elements of R to obtain α-cut of $\tilde{E}$.
Therefore,

$$^{\alpha}\tilde{E} = [4\alpha^2 + 14\alpha + 12,\ 3\alpha^2 - 29\alpha + 56], \quad \forall \alpha \in [0, 1].$$

Using Theorem (2.8), $\tilde{E} = (12, 20, 56)$.
Calculation for original fuzzy number:
Now, we want to obtain original numbers. The following steps are used to obtain $^{\alpha}\tilde{B}$ from $^{\alpha}\tilde{E}$:
Step-1: For $^{\alpha}\tilde{E} \oslash {}^{\alpha}\tilde{A}$, obtain L as,

$$L = \left[\frac{4\alpha^2 + 14\alpha + 12}{2\alpha + 3}, \frac{4\alpha^2 + 14\alpha + 12}{8 - 3\alpha}, \frac{3\alpha^2 - 29\alpha + 56}{2\alpha + 3}, \frac{3\alpha^2 - 29\alpha + 56}{8 - 3\alpha}\right]$$

Step-2:

$$R = \left[\frac{4\alpha^2 + 14\alpha + 12}{8 - 3\alpha}, \frac{4\alpha^2 + 14\alpha + 12}{2\alpha + 3}, \frac{3\alpha^2 - 29\alpha + 56}{8 - 3\alpha}, \frac{3\alpha^2 - 29\alpha + 56}{2\alpha + 3}\right].$$

Step-3: Select the 2^{nd} and 3^{rd} elements of R, which gives

$$^{\alpha}\tilde{B} = \left[\frac{4\alpha^2 + 14\alpha + 12}{2\alpha + 3}, \frac{3\alpha^2 - 29\alpha + 56}{8 - 3\alpha}\right] = [2\alpha + 4, 7 - \alpha].$$

Using Theorem (2.8), $\tilde{B} = (4, 5, 7)$.
Now, we determine $\tilde{A}$ as follows:
Step-1: For $^{\alpha}\tilde{E} \oslash {}^{\alpha}\tilde{B}$, obtain L as

$$L = \left[\frac{4\alpha^2 + 14\alpha + 12}{2\alpha + 4}, \frac{4\alpha^2 + 14\alpha + 12}{7 - \alpha}, \frac{3\alpha^2 - 29\alpha + 56}{2\alpha + 4}, \frac{3\alpha^2 - 29\alpha + 56}{7 - \alpha}\right]$$

Step-2:

$$R = \left[\frac{4\alpha^2+14\alpha+12}{7-\alpha}, \frac{4\alpha^2+14\alpha+12}{2\alpha+4}, \frac{3\alpha^2-29\alpha+56}{7-\alpha}, \frac{3\alpha^2-29\alpha+56}{2\alpha+4}\right].$$

Step-3: Select the 2^{nd} and 3^{rd} elements of R, which gives

$${}^{\alpha}\tilde{A} = \left[\frac{4\alpha^2+14\alpha+12}{2\alpha+4}, \frac{3\alpha^2-29\alpha+56}{7-\alpha}\right] = [2\alpha+3, 8-3\alpha].$$

Using Theorem (2.8), $\tilde{A} = (3, 5, 8)$.

Division:

Let division of $\tilde{A}$ by $\tilde{B}$ be $\tilde{F}, i.e. \tilde{A} \oslash \tilde{B} = \tilde{F}$. To obtain ${}^{\alpha}\tilde{F} =^{\alpha} \tilde{A}/^{\alpha}\tilde{B}$, We perform the following steps:
Step-1:

$$L = [a_l/b_l, a_l/b_u, a_u/b_l, a_u/b_u] = \left[\frac{2\alpha+3}{2\alpha+4}, \frac{2\alpha+3}{7-\alpha}, \frac{8-3\alpha}{2\alpha+4}, \frac{8-3\alpha}{7-\alpha}\right]$$

Step-2: Now apply the fsort() function,

$$R = \left[\frac{2\alpha+3}{7-\alpha}, \frac{2\alpha+3}{2\alpha+4}, \frac{8-3\alpha}{7-\alpha}, \frac{8-3\alpha}{2\alpha+4}\right].$$

Step-3: Select the first and last elements of R to obtain α-cut of $\tilde{F}$.
Therefore,

$${}^{\alpha}\tilde{F} = \left[\frac{2\alpha+3}{7-\alpha}, \frac{8-3\alpha}{2\alpha+4}\right].$$

Using Theorem (2.8), $\tilde{F} = (3/7, 5/6, 2)$.
Calculation for original fuzzy number:
We want to find original numbers from obtained $\tilde{F}$. To compute ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} \otimes {}^{\alpha}\tilde{B}$, Steps are as follows:
Step-1:

$$L = \left[\frac{4\alpha^2+14\alpha+12}{7-\alpha}, 8-3\alpha, 2\alpha+3, \frac{3\alpha^2-29\alpha+56}{2\alpha+4}\right]$$

Step-2: Apply the fsort() function, R will be,

$$R = \left[\frac{4\alpha^2+14\alpha+12}{7-\alpha}, 2\alpha+3, 8-3\alpha, \frac{3\alpha^2-29\alpha+56}{2\alpha+4}\right].$$

Step-3: Select the 2^{nd} and 3^{rd} elements of R, which gives

$${}^{\alpha}\tilde{A} = [2\alpha+3, 8-3\alpha].$$

Using Theorem (2.8), $\tilde{A} = (3, 5, 8)$.
Now, we find ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} \oslash {}^{\alpha}\tilde{F}$ as:
Step-1:

$$L = \left[7-\alpha, \frac{4\alpha^2+14\alpha+12}{8-3\alpha}, \frac{3\alpha^2-29\alpha+56}{2\alpha+3}, 2\alpha+4\right]$$

Step-2: Apply the fsort() function, R will be,

$$R = \left[\frac{4\alpha^2 + 14\alpha + 12}{8 - 3\alpha}, 2\alpha + 4, 7 - \alpha, \frac{3\alpha^2 - 29\alpha + 56}{2\alpha + 3}\right].$$

Step-3: Select the 2^{nd} and 3^{rd} elements of R, which gives

$${}^{\alpha}\tilde{B} = [2\alpha + 4, 7 - \alpha].$$

Using Theorem (2.8), $\tilde{B} = (5, 6, 7)$.

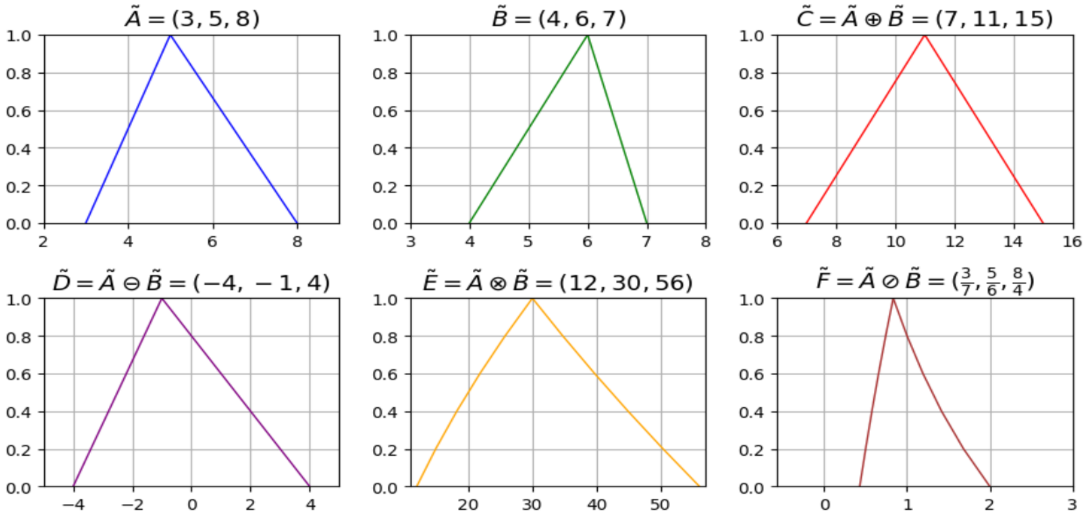

Figure 2: Case : length($\tilde{A}$) > length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are both positive.

$\tilde{A}$ and $\tilde{B}$ are negative TFNs			
$\tilde{A} = (-7, -4, -2)$, ${}^{\alpha}\tilde{A} = [3\alpha - 7, -2 - 2\alpha]$ and $\tilde{B} = (-6, -5, -4)$, ${}^{\alpha}\tilde{B} = [\alpha - 6, -4 - \alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C} = \tilde{A} \oplus \tilde{B}$	$L = [4\alpha - 13, 2\alpha - 11, -8 - \alpha, -6 - 3\alpha]$ $R = [4\alpha - 13, 2\alpha - 11, -8 - \alpha, -6 - 3\alpha]$ ${}^{\alpha}\tilde{C} = [4\alpha - 13, -6 - 3\alpha]$. Using FDT, $\tilde{C} = (-13, -9, -6)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{B}$ $L = [3\alpha - 7, 5\alpha - 9, -4\alpha, -2 - 2\alpha]$ $R = [5\alpha - 9, 3\alpha - 7, -2 - 2\alpha, -4\alpha]$ ${}^{\alpha}\tilde{A} = [3\alpha - 7, -2 - 2\alpha]$. Using FDT, $\tilde{A} = (-7, -4, -2)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{A}$ $L = [\alpha - 6, 6\alpha - 11, 1 - 6\alpha, -4 - \alpha]$ $R = [6\alpha - 11, \alpha - 6, -4 - \alpha, 1 - 6\alpha]$ ${}^{\alpha}\tilde{B} = [\alpha - 6, -4 - \alpha]$. Using FDT, $\tilde{B} = (-6, -5, -4)$.
(ii) $\tilde{D} = \tilde{A} \ominus \tilde{B}$	$L = [2\alpha - 1, 4\alpha - 3, 4 - 3\alpha, 2 - \alpha]$ $R = [4\alpha - 3, 2\alpha - 1, 2 - \alpha, 4 - 3\alpha]$ ${}^{\alpha}\tilde{D} = [4\alpha - 3, 4 - 3\alpha]$. Using FDT, $\tilde{D} = (-3, 1, 4)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{D} + {}^{\alpha}\tilde{B}$ $L = [5\alpha - 9, 3\alpha - 7, -2 - 2\alpha, -4\alpha]$ $R = [5\alpha - 9, 3\alpha - 7, -2 - 2\alpha, -4\alpha]$ ${}^{\alpha}\tilde{A} = [3\alpha - 7, -2 - 2\alpha]$. Using FDT, $\tilde{A} = (-7, -4, -2)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} - {}^{\alpha}\tilde{D}$ $L = [-4 - \alpha, 6\alpha - 11, 1 - 6\alpha, \alpha - 6]$ $R = [6\alpha - 11, \alpha - 6, -4 - \alpha, 1 - 6\alpha]$ ${}^{\alpha}\tilde{B} = [\alpha - 6, -4 - \alpha]$. Using FDT, $\tilde{B} = (-6, -5, -4)$.
(iii) $\tilde{E} = \tilde{A} \otimes \tilde{B}$	$L = [3\alpha^2 - 25\alpha + 42, -3\alpha^2 - 5\alpha + 28$ $-2\alpha^2 + 10\alpha + 12, 2\alpha^2 + 10\alpha + 8]$ $R = [2\alpha^2 + 10\alpha + 8, -2\alpha^2 + 10\alpha + 12,$ $-3\alpha^2 - 5\alpha + 28, 3\alpha^2 - 25\alpha + 42]$ ${}^{\alpha}\tilde{E} = [2\alpha^2 + 10\alpha + 8, 3\alpha^2 - 25\alpha + 42]$. Using FDT, $\tilde{E} = (8, 20, 42)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{E} / {}^{\alpha}\tilde{B}$ $L = [2\alpha^2 + 10\alpha + 8/\alpha - 6,$ $2\alpha^2 + 10\alpha + 8/-4 - \alpha,$ $3\alpha^2 - 25\alpha + 42/\alpha - 6,$ $3\alpha^2 - 25\alpha + 42/-4 - \alpha]$ $R = [3\alpha^2 - 25\alpha + 42/-4 - \alpha,$ $3\alpha^2 - 25\alpha + 42/\alpha - 6,$ $2\alpha^2 + 10\alpha + 8/-4 - \alpha,$ $2\alpha^2 + 10\alpha + 8/\alpha - 6]$ ${}^{\alpha}\tilde{A} = [3\alpha - 7, -2 - 2\alpha]$. Using FDT, $\tilde{A} = (-7, -4, -2)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{E} / {}^{\alpha}\tilde{A}$ $L = [2\alpha^2 + 10\alpha + 8/3\alpha - 7,$ $2\alpha^2 + 10\alpha + 8/-2 - 2\alpha,$ $3\alpha^2 - 25\alpha + 42/3\alpha - 7,$ $3\alpha^2 - 25\alpha + 42/-2 - 2\alpha]$ $R = [3\alpha^2 - 25\alpha + 42/-2 - 2\alpha,$ $3\alpha^2 - 25\alpha + 42/3\alpha - 7,$ $2\alpha^2 + 10\alpha + 8/-2 - 2\alpha,$ $2\alpha^2 + 10\alpha + 8/3\alpha - 7]$ ${}^{\alpha}\tilde{B} = [\alpha - 6, -4 - \alpha]$. Using FDT, $\tilde{B} = (-6, -5, -4)$.

Arithmetic operation	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} / {}^{\alpha}\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [3\alpha - 7/\alpha - 6, 3\alpha - 7/-4-\alpha, -2-2\alpha/\alpha-6, -2-2\alpha/-4-\alpha]$ $R = [-2-2\alpha/\alpha-6, -2-2\alpha/-4-\alpha, 3\alpha-7/\alpha-6, 3\alpha-7/-4-\alpha]$ ${}^{\alpha}\tilde{F} = [-2-2\alpha/\alpha-6, 3\alpha-7/-4-\alpha]^{\alpha}$. Using FDT, $\tilde{F} = (1/3, 4/5, 7/4)$.	$L = [-2-2\alpha, 2\alpha^2+10\alpha+8/\alpha-6, 3\alpha^2-25\alpha+42/-4-\alpha, 3\alpha-7]$ $R = [3\alpha^2-25\alpha+42/-4-\alpha, 3\alpha-7-2-2\alpha, 2\alpha^2+10\alpha+8/\alpha-6]$ $\tilde{A} = [3\alpha-7, -2-2\alpha]$. Using FDT, $\tilde{A} = (-7, -4, -2)$.	$L = [3\alpha^2-25\alpha+42/-2-2\alpha, -4-\alpha, \alpha-6, 2\alpha^2+10\alpha+8/3\alpha-7]$ $,R = [3\alpha^2-25\alpha+42/-2-2\alpha, \alpha-6, -4-\alpha, 2\alpha^2+10\alpha+8/3\alpha-7]$ ${}^{\alpha}\tilde{B} = [\alpha-6, -4-\alpha]$. Using FDT, $\tilde{B} = (-6, -5, -4)$.

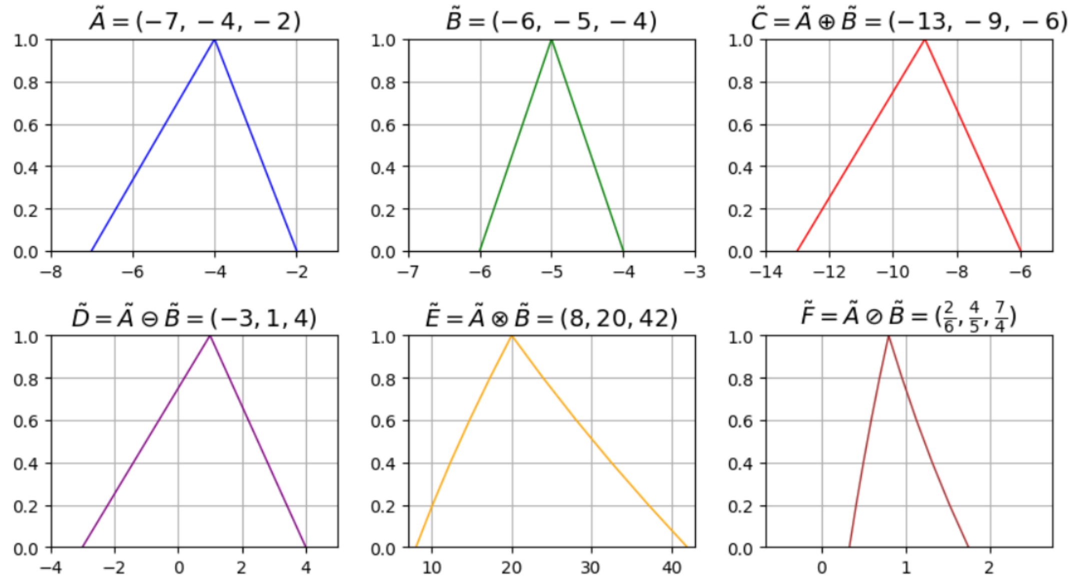

Figure 3: Case: length($\tilde{A}$) > length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are negative TFNs.

$\tilde{A}$ is positive and $\tilde{B}$ is negative TFNs			
$\tilde{A} = (1,3,4)$, ${}^{\alpha}\tilde{A} = [2\alpha+1, 4-\alpha]$ and $\tilde{B} = (-3,-2,-1)$, ${}^{\alpha}\tilde{B} = [\alpha-3, -1-\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C} = \tilde{A} \oplus \tilde{B}$	$L = [3\alpha-2, \alpha, 1, 3-2\alpha]$ $R = [3\alpha-2, \alpha, 1, 3-2\alpha]$ ${}^{\alpha}\tilde{C} = [3\alpha-2, 3-2\alpha]$. Using FDT, $\tilde{C} = (-2,1,3)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{B}$ $L = [2\alpha+1, 4\alpha-1, 6-3\alpha, 4-\alpha]$ $R = [4\alpha-1, 2\alpha+1, 4-\alpha, 6-3\alpha]$ ${}^{\alpha}\tilde{A} = [2\alpha+1, 4-\alpha]$. Using FDT, $\tilde{A} = (1,3,4)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{A}$ $L = [\alpha-3, 4\alpha-6, 1-4\alpha, -1-\alpha]$ $R = [4\alpha-1, \alpha-3, -1-\alpha, 1-4\alpha]$ ${}^{\alpha}\tilde{B} = [\alpha-3, -1-\alpha]$. Using FDT, $\tilde{B} = (-3,-2,-1)$.
(ii) $\tilde{D} = \tilde{A} \ominus \tilde{B}$	$L = [\alpha+4, 3\alpha+2, 7-2\alpha, 5]$ $R = [3\alpha+2, \alpha+4, 5, 7-2\alpha]$ ${}^{\alpha}\tilde{D} = [3\alpha+2, 7-2\alpha]$. Using FDT, $\tilde{D} = (2,5,7)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{D} + {}^{\alpha}\tilde{B}$ $L = [4\alpha-1, 2\alpha+1, 4-\alpha, 6-3\alpha]$ $R = [4\alpha-1, 2\alpha+1, 4-\alpha, 6-3\alpha]$ ${}^{\alpha}\tilde{A} = [2\alpha+1, 4-\alpha,]$. Using FDT, $\tilde{A} = (1,3,4)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} - {}^{\alpha}\tilde{D}$ $L = [-1-\alpha, 4\alpha-6, 2-4\alpha, \alpha-3]$ $R = [4\alpha-6, \alpha-3, -1-\alpha, 2-4\alpha]$ ${}^{\alpha}\tilde{B} = [\alpha-3, -1-\alpha]$. Using FDT, $\tilde{B} = (-3,-2,-1)$.
(iii) $\tilde{E} = \tilde{A} \otimes \tilde{B}$	$L = [2\alpha^2-5\alpha-3, -2\alpha^2-3\alpha-1,$ $-\alpha^2+7\alpha-12, \alpha^2-3\alpha-4]$ $R = [-\alpha^2+7\alpha-12, \alpha^2-3\alpha-4,$ $2\alpha^2-5\alpha-3, -2\alpha^2-3\alpha-1]$ ${}^{\alpha}\tilde{E} = [-\alpha^2+7\alpha-12, -2\alpha^2-3\alpha-1]$. Using FDT, $\tilde{E} = (-12,-6,-1)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L = [-\alpha^2+7\alpha-12/\alpha-3,$ $-\alpha^2+7\alpha-12/-1-\alpha,$ $-2\alpha^2-3\alpha-1/\alpha-3,$ $-2\alpha^2-3\alpha-1/-1-\alpha]$ $R = [\ -2\alpha^2-3\alpha-1/\alpha-3,$ $-2\alpha^2-3\alpha-1/-1-\alpha,$ $-\alpha^2+7\alpha-12/\alpha-3,$ $-\alpha^2+7\alpha-12/-1-\alpha]$ ${}^{\alpha}\tilde{A} = [2\alpha+1, 4-\alpha]$. Using FDT, $\tilde{A} = (1,3,4)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L = [-\alpha^2+7\alpha-12/2\alpha+1,$ $-\alpha^2+7\alpha-12/4-\alpha,$ $-2\alpha^2-3\alpha-1/2\alpha+1,$ $-2\alpha^2-3\alpha-1/4-\alpha]$ $R = [-\alpha^2+7\alpha-12/2\alpha+1,$ $-\alpha^2+7\alpha-12/4-\alpha,$ $-2\alpha^2-3\alpha-1/2\alpha+1,$ $-2\alpha^2-3\alpha-1/4-\alpha]$ ${}^{\alpha}\tilde{B} = [\alpha-3, -1-\alpha]$. Using FDT, $\tilde{B} = (-3,-2,-1)$.

Arithmetic operations	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{C} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A}/{}^{\alpha}\tilde{C}$
(iv) $\tilde{C} = \tilde{A} \oslash \tilde{B}$	$L = [2\alpha + 1/\alpha - 3, 2\alpha + 1/-1-\alpha, 4-\alpha/\alpha-3, 4-\alpha/-1-\alpha]$ $R = [4-\alpha/-1-\alpha, 4-\alpha/\alpha-3, 2\alpha+1/-1-\alpha, 2\alpha+1/\alpha-3]$ ${}^{\alpha}\tilde{C} = [4-\alpha/-1-\alpha, 2\alpha+1/\alpha-3]$. Using FDT, $\tilde{C} = (-4, -3/2, -1/3)$.	$L = [-\alpha^2 + 7\alpha - 12/-1-\alpha, 4-\alpha, 2\alpha+1, -2\alpha^2 - 3\alpha - 1/\alpha - 3]$ $R = [-2\alpha^2 - 3\alpha - 1/\alpha - 3, 4-\alpha, 2\alpha+1, -\alpha^2 + 7\alpha - 12/-1-\alpha]$ ${}^{\alpha}\tilde{A} = [2\alpha+1, 4-\alpha]$. Using FDT, $\tilde{A} = (1, 3, 4)$.	$L = [-2\alpha^2 - 3\alpha - 1/4-\alpha, \alpha-3, -1-\alpha, -\alpha^2 + 7\alpha - 12/2\alpha+1]$ $R = [-\alpha^2 + 7\alpha - 12/2\alpha+1, \alpha-3, -1-\alpha, -2\alpha^2 - 3\alpha - 1/4-\alpha]$ ${}^{\alpha}\tilde{B} = [\alpha-3, -1-\alpha]$. Using FDT, $\tilde{B} = (-3, -2, -1)$.

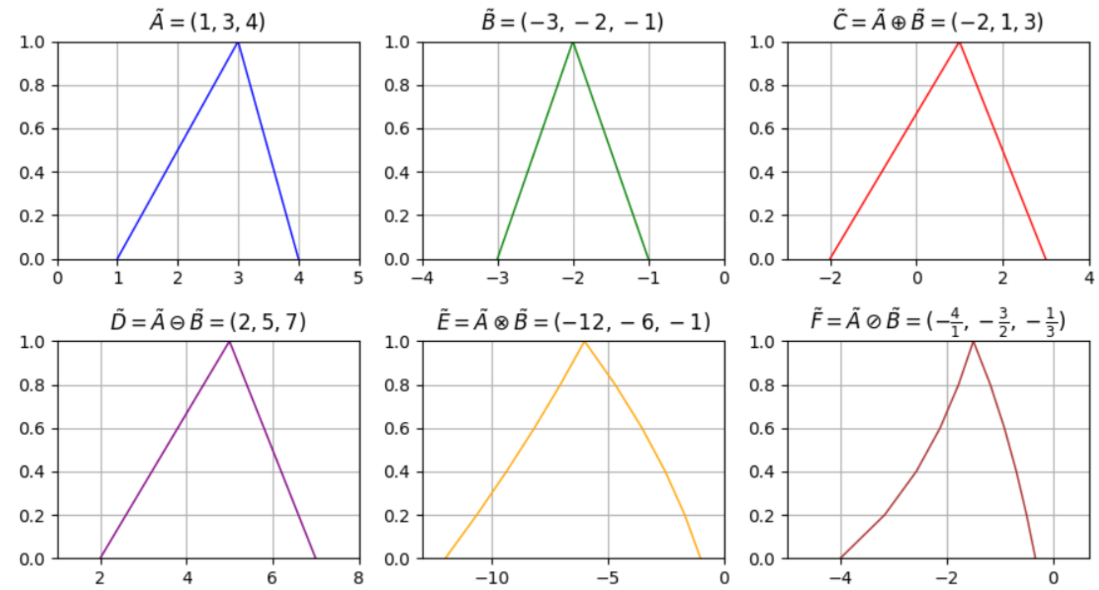

Figure 4: Case: length($\tilde{A}$) > length($\tilde{B}$), considering $\tilde{A}$ is positive $\tilde{B}$ is negative TFNs.

$\tilde{A}$ and $\tilde{B}$ are around zero TFNs			
$\tilde{A}=(-3,-1,3)$, ${}^{\alpha}\tilde{A}=[2\alpha-3,3-4\alpha]$ and $\tilde{B}=(-4,-3,1)$, ${}^{\alpha}\tilde{B}=[\alpha-4,1-4\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C}=\tilde{A}\oplus\tilde{B}$	$L=[3\alpha-7,-2\alpha-2,-1-3\alpha,4-8\alpha]$ $R=[3\alpha-7,-2\alpha-2,-1-3\alpha,4-8\alpha]$ ${}^{\alpha}\tilde{C}=[3\alpha-7,4-8\alpha]$. Using FDT, $\tilde{C}=(-7,-4,4)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{B}$ $L=[2\alpha-3,7\alpha-6,8-9\alpha,3-4\alpha]$ $R=[7\alpha-6,2\alpha-3,3-4\alpha,8-9\alpha]$ ${}^{\alpha}\tilde{A}=[2\alpha-3,3-4\alpha]$. Using FDT, $\tilde{A}=(-3,-1,3)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{A}$ $L=[7\alpha-10,\alpha-4,1-4\alpha,7-10\alpha]$ $R=[7\alpha-10,\alpha-4,1-4\alpha,7-10\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-4,1-4\alpha]$. Using FDT, $\tilde{B}=(-4,-3,1)$.
(ii) $\tilde{D}=\tilde{A}\ominus\tilde{B}$	$L=[\alpha+1,6\alpha-4,7-5\alpha,2]$ $R=[6\alpha-4,\alpha+1,2,7-5\alpha]$ ${}^{\alpha}\tilde{D}=[6\alpha-4,7-5\alpha]$. Using FDT, $\tilde{D}=(-4,2,7)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{D}+{}^{\alpha}\tilde{B}$ $L=[7\alpha-8,2\alpha-3,3-4\alpha,8-9\alpha]$ $R=[7\alpha-8,2\alpha-3,3-4\alpha,8-9\alpha]$ ${}^{\alpha}\tilde{A}=[2\alpha-3,3-4\alpha]$. Using FDT, $\tilde{A}=(-3,-1,3)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{A}-{}^{\alpha}\tilde{D}$ $L=[1-4\alpha,7\alpha-10,7-10\alpha,\alpha-4]$ $R=[7\alpha-10,\alpha-4,1-4\alpha,7-10\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-4,1-4\alpha]$. Using FDT, $\tilde{B}=(-4,-3,1)$.

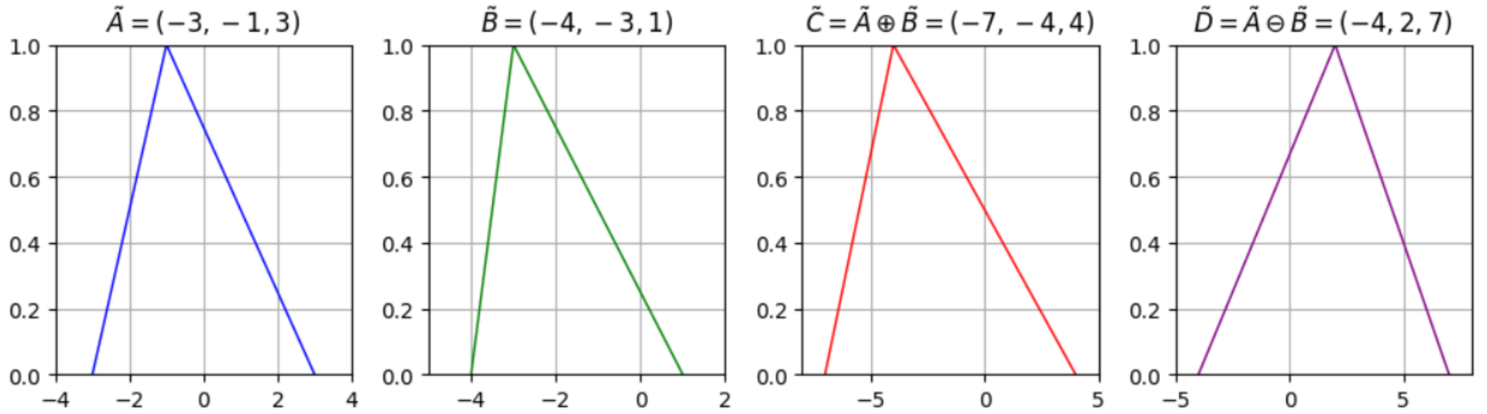

Figure 5: Case: length($\tilde{A}$) > length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are around zero TFNs.

Case 2: length($\tilde{A}$)< length($\tilde{B}$)			
$\tilde{A}$ and $\tilde{B}$ are positive TFNs			
$\tilde{A}=(1,2,3)$, ${}^{\alpha}\tilde{A}=[\alpha+1,3-\alpha]$ and $\tilde{B}=(7,9,11)$, ${}^{\alpha}\tilde{B}=[2\alpha+7,11-2\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C}=\tilde{A}\oplus\tilde{B}$	$L=[3\alpha+8,12-\alpha,10+\alpha,14-3\alpha]$ $R=[3\alpha+8,10+\alpha,12-\alpha,14-3\alpha]$ ${}^{\alpha}\tilde{C}=[3\alpha+8,14-3\alpha]$. Using FDT, $\tilde{C}=(8,11,14)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{B}$ $L=[\alpha+1,5\alpha-3,7-5\alpha,3-\alpha]$ $R=[5\alpha-3,\alpha+1,3-\alpha,7-5\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+1,3-\alpha]$. Using FDT, $\tilde{A}=(1,2,3)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{A}$ $L=[2\alpha+7,4\alpha+5,13-4\alpha,11-2\alpha]$ $R=[4\alpha+5,2\alpha+7,11-2\alpha,13-4\alpha]$ ${}^{\alpha}\tilde{B}=[2\alpha+7,11-2\alpha]$. Using FDT, $\tilde{B}=(7,9,11)$.
(ii) $\tilde{D}=\tilde{A}\ominus\tilde{B}$	$L=[-\alpha-6,3\alpha-10,-3\alpha-4,\alpha-8]$ $R=[3\alpha-10,\alpha-8,-\alpha-6,-3\alpha-4]$ ${}^{\alpha}\tilde{D}=[3\alpha-10,-3\alpha-4]$. Using FDT, $\tilde{D}=(-10,-7,-4)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{D}+{}^{\alpha}\tilde{B}$ $L=[5\alpha-3,\alpha+1,3-\alpha,7-5\alpha]$ $R=[5\alpha-3,\alpha+1,3-\alpha,7-5\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+1,3-\alpha]$. Using FDT, $\tilde{A}=(1,2,3)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{A}-{}^{\alpha}\tilde{D}$ $L=[11-2\alpha,4\alpha+5,13-4\alpha,7+2\alpha]$ $R=[4\alpha+5,7+2\alpha,11-2\alpha,13-4\alpha]$ ${}^{\alpha}\tilde{B}=[7+2\alpha,11-2\alpha]$. Using FDT, $\tilde{B}=(7,9,11)$.
(iii) $\tilde{E}=\tilde{A}\otimes\tilde{B}$	$L=[2\alpha^2+9\alpha+7,-2\alpha^2+9\alpha+11$ $-2\alpha^2-\alpha+21,2\alpha^2-17\alpha+33]$ $R=[2\alpha^2+9\alpha+7,-2\alpha^2+9\alpha+11$ $-2\alpha^2-\alpha+21,2\alpha^2-17\alpha+33]$ ${}^{\alpha}\tilde{E}=[2\alpha^2+9\alpha+7,2\alpha^2-17\alpha+33]$. Using FDT, $\tilde{E}=(7,18,33)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L=[2\alpha^2+9\alpha+7/2\alpha+7,$ $2\alpha^2+9\alpha+7/11-2\alpha,$ $2\alpha^2-17\alpha+33/2\alpha+7,$ $2\alpha^2-17\alpha+33/11-2\alpha]$ $R=[2\alpha^2+9\alpha+7/11-2\alpha,$ $2\alpha^2+9\alpha+7/2\alpha+7,$ $2\alpha^2-17\alpha+33/2\alpha+7,$ $2\alpha^2-17\alpha+33/11-2\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+1,3-\alpha]$. Using FDT, $\tilde{A}=(1,2,3)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L=[2\alpha^2+9\alpha+7/\alpha+1,$ $2\alpha^2+9\alpha+7/3-\alpha,$ $2\alpha^2-17\alpha+33/\alpha+1,$ $2\alpha^2-17\alpha+33/3-\alpha]$ $R=[2\alpha^2+9\alpha+7/3-\alpha,$ $2\alpha^2+9\alpha+7/\alpha+1,$ $2\alpha^2-17\alpha+33/3-\alpha,$ $2\alpha^2-17\alpha+33/\alpha+1]$ ${}^{\alpha}\tilde{B}=[2\alpha+7,11-2\alpha]$. Using FDT, $\tilde{B}=(7,9,11)$.

Arithmetic operations	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} / {}^{\alpha}\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [\alpha + 1/2\alpha + 7, \alpha + 1/11 - 2\alpha, 3 - \alpha/2\alpha + 7, 3 - \alpha/11 - 2\alpha]$ $R = [\alpha + 1/11 - 2\alpha, \alpha + 1/2\alpha + 7, 3 - \alpha/11 - 2\alpha, 3 - \alpha/2\alpha + 7]$ ${}^{\alpha}\tilde{F} = [\alpha + 1/11 - 2\alpha, 3 - \alpha/2\alpha + 7]$. Using FDT, $\tilde{F} = (1/11, 2/9, 3/7)$.	$L = [2\alpha^2 + 9\alpha + 7/11 - 2\alpha, \alpha + 1, 3 - \alpha, 2\alpha^2 - 17\alpha + 33/2\alpha + 7]$ $R = [2\alpha^2 + 9\alpha + 7/11 - 2\alpha, \alpha + 1, 3 - \alpha, 2\alpha^2 - 17\alpha + 33/2\alpha + 7]$ ${}^{\alpha}\tilde{A} = [\alpha + 1, 3 - \alpha]$. Using FDT, $\tilde{A} = (1, 2, 3)$.	$L = [11 - 2\alpha, 2\alpha^2 + 9\alpha + 7/3 - \alpha, 2\alpha^2 - 17\alpha + 33/\alpha + 1, 2\alpha + 7]$ $R = [2\alpha^2 + 9\alpha + 7/3 - \alpha, 2\alpha + 7, 11 - 2\alpha, 2\alpha^2 - 17\alpha + 33/\alpha + 1]$ ${}^{\alpha}\tilde{B} = [2\alpha + 7, 11 - 2\alpha]$. Using FDT, $\tilde{B} = (7, 9, 11)$.

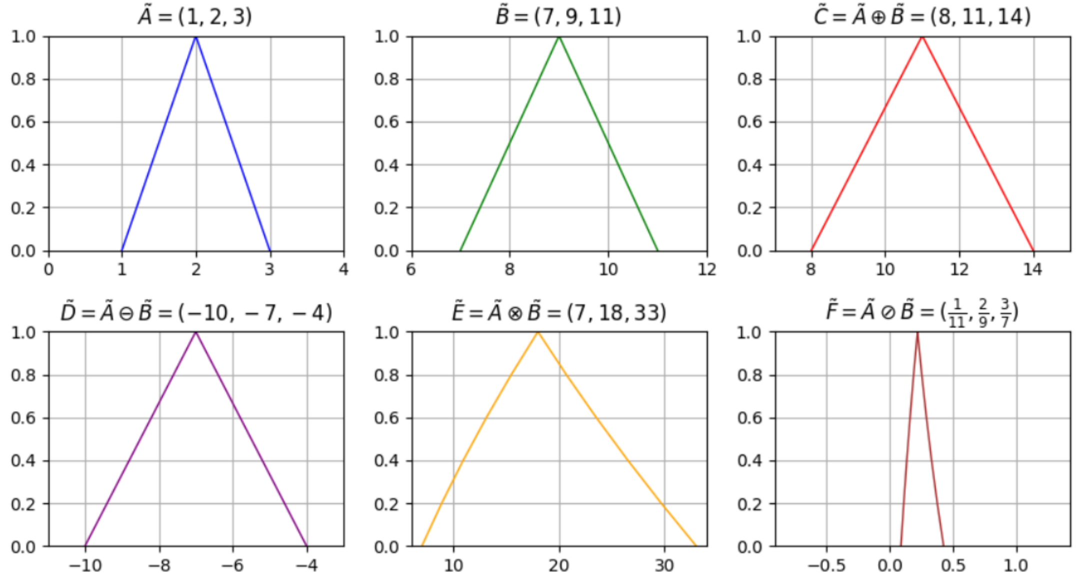

Figure 6: Case: length($\tilde{A}$) < length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are positive TFNs.

$\tilde{A}$ and $\tilde{B}$ are negative TFNs			
$\tilde{A} = (-5, -3, -2)$, ${}^{\alpha}\tilde{A} = [2\alpha - 5, -2 - \alpha]$ and $\tilde{B} = (-7, -4, -1)$, ${}^{\alpha}\tilde{B} = [3\alpha - 7, -1 - 3\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C} = \tilde{A} \oplus \tilde{B}$	$L = [5\alpha - 12, -\alpha - 6, 2\alpha - 9, -3 - 4\alpha]$ $R = [5\alpha - 12, 2\alpha - 9, -\alpha - 6, -3 - 4\alpha]$ ${}^{\alpha}\tilde{C} = [5\alpha - 12, -3 - 4\alpha]$. Using FDT, $\tilde{C} = (-12, -7, -3)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{B}$ $L = [2\alpha - 5, 8\alpha - 11, 4 - 7\alpha, -2 - \alpha]$ $R = [8\alpha - 11, 2\alpha - 5, -2 - \alpha, 4 - 7\alpha]$ ${}^{\alpha}\tilde{A} = [2\alpha - 5, -2 - \alpha]$. Using FDT, $\tilde{A} = (-5, -3, -2)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{A}$ $L = [3\alpha - 7, 6\alpha - 14, 2 - 6\alpha, -1 - 3\alpha]$ $R = [6\alpha - 14, 3\alpha - 7, -1 - 3\alpha, 2 - 6\alpha]$ ${}^{\alpha}\tilde{B} = [3\alpha - 7, -1 - 3\alpha]$. Using FDT, $\tilde{B} = (-7, -4, -1)$.
(ii) $\tilde{D} = \tilde{A} \ominus \tilde{B}$	$L = [2 - \alpha, 5\alpha - 4, 5 - 4\alpha, 2\alpha - 1]$ $R = [5\alpha - 4, 2\alpha - 1, 2 - \alpha, 5 - 4\alpha]$ ${}^{\alpha}\tilde{D} = [5\alpha - 4, 5 - 4\alpha]$. Using FDT, $\tilde{D} = (-4, 1, 5)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{D} + {}^{\alpha}\tilde{B}$ $L = [8\alpha - 11, 2\alpha - 5, -2 - \alpha, 6 - 7\alpha]$ $R = [8\alpha - 11, 2\alpha - 5, -2 - \alpha, 6 - 7\alpha]$ ${}^{\alpha}\tilde{A} = [2\alpha - 5, -2 - \alpha]$. Using FDT, $\tilde{A} = (-5, -3, -2)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} - {}^{\alpha}\tilde{D}$ $L = [-1 - 3\alpha, 6\alpha - 10, 2 + 6\alpha, 3\alpha - 7]$ $R = [6\alpha - 10, 3\alpha - 7, -1 - 3\alpha, 2 - 6\alpha]$ ${}^{\alpha}\tilde{B} = [3\alpha - 7, -1 - 3\alpha]$. Using FDT, $\tilde{B} = (-7, -4, -1)$.
(iii) $\tilde{E} = \tilde{A} \otimes \tilde{B}$	$L = [6\alpha^2 - 29\alpha + 35, -6\alpha^2 + 13\alpha + 5$ $- 3\alpha^2 + \alpha + 14, 3\alpha^2 + 7\alpha + 2]$ $R = [3\alpha^2 + 7\alpha + 2, -6\alpha^2 + 13\alpha + 5,$ $- 3\alpha^2 + \alpha + 14, 6\alpha^2 - 29\alpha + 35]$ ${}^{\alpha}\tilde{E} = [3\alpha^2 + 7\alpha + 2, 6\alpha^2 - 29\alpha + 35]$. Using FDT, $\tilde{E} = (2, 12, 35)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L = [3\alpha^2 + 7\alpha + 2/2\alpha - 5,$ $3\alpha^2 + 7\alpha + 2/-2 - \alpha,$ $6\alpha^2 - 29\alpha + 35/2\alpha - 5,$ $6\alpha^2 - 29\alpha + 35/-2 - \alpha]$ $R = [6\alpha^2 - 29\alpha + 35/-2 - \alpha,$ $6\alpha^2 - 29\alpha + 35/2\alpha - 5,$ $3\alpha^2 + 7\alpha + 2/-2 - \alpha,$ $3\alpha^2 + 7\alpha + 2/2\alpha - 5]$ ${}^{\alpha}\tilde{A} = [3\alpha - 7, -1 - 3\alpha]$. Using FDT, $\tilde{A} = (-7, -4, -1)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L = [3\alpha^2 + 7\alpha + 2/3\alpha - 7,$ $3\alpha^2 + 7\alpha + 2/-1 - 3\alpha,$ $6\alpha^2 - 29\alpha + 35/3\alpha - 7,$ $6\alpha^2 - 29\alpha + 35/-1 - 3\alpha]$ $R = [6\alpha^2 - 29\alpha + 35/-1 - 3\alpha,$ $6\alpha^2 - 29\alpha + 35/3\alpha - 7,$ $3\alpha^2 + 7\alpha + 2/-1 - 3\alpha,$ $3\alpha^2 + 7\alpha + 2/3\alpha - 7]$ ${}^{\alpha}\tilde{B} = [2\alpha - 5, -2 - \alpha]$. Using FDT, $\tilde{B} = (-5, -3, -2)$.

Arithmetic operation	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A}/{}^{\alpha}\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [2\alpha - 5/3\alpha - 7, 2\alpha - 5/-1-3\alpha, -2-\alpha/3\alpha - 7, -2-\alpha/-1-3\alpha]$ $R = [-2-\alpha/3\alpha-7, 2\alpha-5/3\alpha-7, -2-\alpha/-1-3\alpha, 2\alpha-5/-1-3\alpha]$ ${}^{\alpha}\tilde{F} = [-2-\alpha/3\alpha-7, 2\alpha-5/-1-3\alpha]$. Using FDT, $\tilde{F} = (2/7, 3/4, 5)$.	$L = [-2-\alpha, 3\alpha^2 - 5\alpha - 2/3\alpha - 7, 6\alpha^2 - 29\alpha + 35/-1-3\alpha, 2\alpha - 5]$ $R = [6\alpha^2 - 29\alpha + 35/-1-3\alpha, 2\alpha - 5, -2-\alpha, 3\alpha^2 - 5\alpha - 2/3\alpha - 7]$ ${}^{\alpha}\tilde{A} = [2\alpha - 5, -2-\alpha]$. Using FDT, $\tilde{A} = (-5, -3, -2)$.	$L = [6\alpha^2 - 29\alpha + 35/-2-\alpha, -1-3\alpha, 3\alpha - 7, 3\alpha^2 + 7\alpha + 2/2\alpha - 5]$ $R = [6\alpha^2 - 29\alpha + 35/-2-\alpha, 3\alpha - 7, -1-3\alpha, 3\alpha^2 + 7\alpha + 2/2\alpha - 5]$ ${}^{\alpha}\tilde{B} = [3\alpha - 7, -1-3\alpha]$. Using FDT, $\tilde{B} = (-7, -4, -1)$.

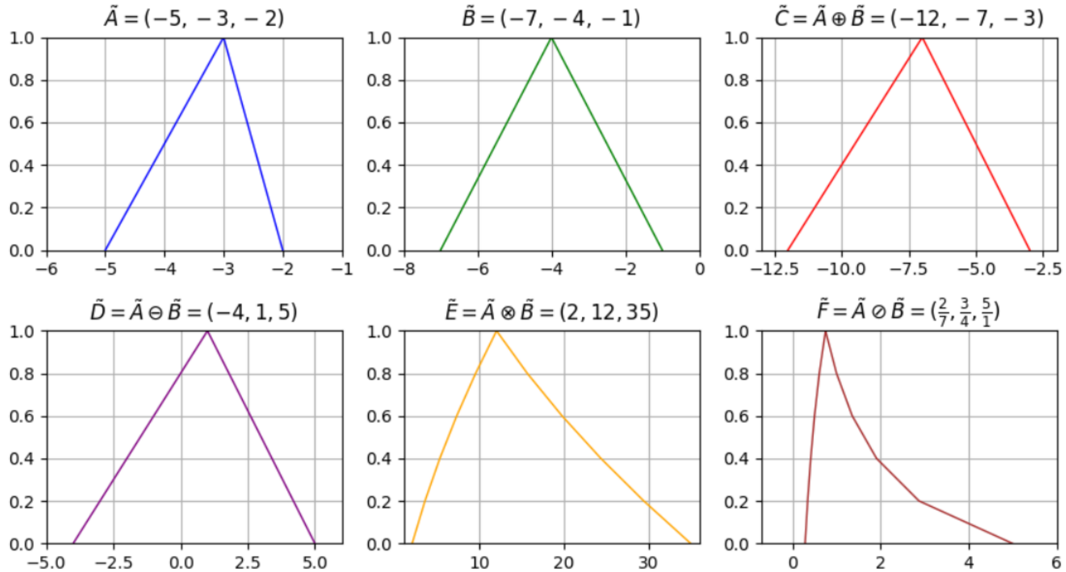

Figure 7: Case: length($\tilde{A}$) < length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are negative TFNs.

$\tilde{A}$ is negative and $\tilde{B}$ is positive TFNs			
$\tilde{A}=(-3,-2,-1)$, ${}^{\alpha}\tilde{A}=[\alpha-3,-1-\alpha]$ and $\tilde{B}=(1,3,4)$, ${}^{\alpha}\tilde{B}=[2\alpha+1,4-\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C}=\tilde{A}\oplus\tilde{B}$	$L=[3\alpha-2,1,\alpha,3-2\alpha]$ $R=[3\alpha-2,\alpha,0,3-2\alpha]$ ${}^{\alpha}\tilde{C}=[3\alpha-2,3-2\alpha]$. Using FDT, $\tilde{C}=(-2,1,3)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{B}$ $L=[\alpha-3,4\alpha-6,2-4\alpha,-1-\alpha]$ $R=[4\alpha-6,\alpha-3,-1-\alpha,2-4\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha-3,-1-\alpha]$. Using FDT, $\tilde{A}=(-3,-2,-1)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{A}$ $L=[2\alpha+1,4\alpha-1,6-3\alpha,4-\alpha]$ $R=[4\alpha-1,2\alpha+1,4-\alpha,6-3\alpha]$ ${}^{\alpha}\tilde{B}=[2\alpha+1,4-\alpha]$. Using FDT, $\tilde{B}=(1,3,4)$.
(ii) $\tilde{D}=\tilde{A}\ominus\tilde{B}$	$L=[-\alpha-4,2\alpha-7,-2-3\alpha,-5]$ $R=[2\alpha-7,-5,-\alpha-4,-2-3\alpha]$ ${}^{\alpha}\tilde{D}=[2\alpha-7,-2-3\alpha]$. Using FDT, $\tilde{D}=(-7,-5,-2)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{D}+{}^{\alpha}\tilde{B}$ $L=[4\alpha-6,\alpha-3,-1-\alpha,2-4\alpha]$ $R=[4\alpha-6,\alpha-3,-1-\alpha,2-4\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha-3,-1-\alpha]$. Using FDT, $\tilde{A}=(-3,-2,-1)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{A}-{}^{\alpha}\tilde{D}$ $L=[4-\alpha,4\alpha-1,6-3\alpha,1+2\alpha]$ $R=[4\alpha-1,1+2\alpha,4-\alpha,6-3\alpha]$ ${}^{\alpha}\tilde{B}=[1+2\alpha,4-\alpha]$. Using FDT, $\tilde{B}=(1,3,4)$.
(iii) $\tilde{E}=\tilde{A}\otimes\tilde{B}$	$L=[2\alpha^2-5\alpha-3,-\alpha^2+7\alpha-12,$ $-2\alpha^2-3\alpha-1,\alpha^2-3\alpha-4]$ $R=[-\alpha^2+7\alpha-12,\alpha^2-3\alpha-4,$ $2\alpha^2-5\alpha-3,-2\alpha^2-3\alpha-1]$ ${}^{\alpha}\tilde{E}=[-\alpha^2+7\alpha-12,-2\alpha^2-3\alpha-1]$. Using FDT, $\tilde{E}=(-12,-6,-1)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L=[-\alpha^2+7\alpha-12/2\alpha+1,$ $-\alpha^2+7\alpha-12/4-\alpha,$ $-2\alpha^2-3\alpha-1/2\alpha+1,$ $-2\alpha^2-3\alpha-1/4-\alpha]$ $R=[-\alpha^2+7\alpha-12/2\alpha+1,$ $-\alpha^2+7\alpha-12/4-\alpha,$ $-2\alpha^2-3\alpha-1/2\alpha+1,$ $-2\alpha^2-3\alpha-1/4-\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha-3,-1-\alpha]$. Using FDT, $\tilde{A}=(-3,-2,-1)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L=[-\alpha^2+7\alpha-12/\alpha-3,$ $-\alpha^2+7\alpha-12/-1-\alpha,$ $-2\alpha^2-3\alpha-1/\alpha-3,$ $-2\alpha^2-3\alpha-1/-1-\alpha]$ $R=[-2\alpha^2-3\alpha-1/\alpha-3,$ $-2\alpha^2-3\alpha-1/-1-\alpha,$ $-\alpha^2+7\alpha-12/\alpha-3,$ $-\alpha^2+7\alpha-12/-1-\alpha]$ ${}^{\alpha}\tilde{B}=[2\alpha+1,4-\alpha]$. Using FDT, $\tilde{B}=(1,3,4)$.

Arithmetic operations	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A}/{}^{\alpha}\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [\alpha - 3/2\alpha + 1, \alpha - 3/4 - \alpha, -1 - \alpha/2\alpha + 1, -1 - \alpha/4 - \alpha]$ $R = [\alpha - 3/2\alpha + 1, -1 - \alpha/2\alpha + 1, \alpha - 3/4 - \alpha, -1 - \alpha/4 - \alpha]$ ${}^{\alpha}\tilde{F} = [\alpha - 3/2\alpha + 1, -1 - \alpha/4 - \alpha]$. Using FDT, $\tilde{F} = (-3, -2/3, -1/4)$.	$L = [\alpha - 3, -\alpha^2 + 7\alpha - 12/2\alpha + 1, -2\alpha^2 - 3\alpha - 1/4 - \alpha, -1 - \alpha]$ $R = [-\alpha^2 + 7\alpha - 12/2\alpha + 1, \alpha - 3, -1 - \alpha, -2\alpha^2 - 3\alpha - 1/4 - \alpha]$ ${}^{\alpha}\tilde{A} = [\alpha - 3, -1 - \alpha]$. Using FDT, $\tilde{A} = (-3, -2, -1)$.	$L = [2\alpha + 1, -\alpha^2 + 7\alpha - 12/ - 1 - \alpha, -2\alpha^2 - 3\alpha - 1/\alpha - 3, 4 - \alpha]$ $R = [-2\alpha^2 - 3\alpha - 1/\alpha - 3, 2\alpha + 1, 4 - \alpha, -\alpha^2 + 7\alpha - 12/ - 1 - \alpha]$ ${}^{\alpha}\tilde{B} = [2\alpha + 1, 4 - \alpha]$. Using FDT, $\tilde{B} = (1, 3, 4)$.

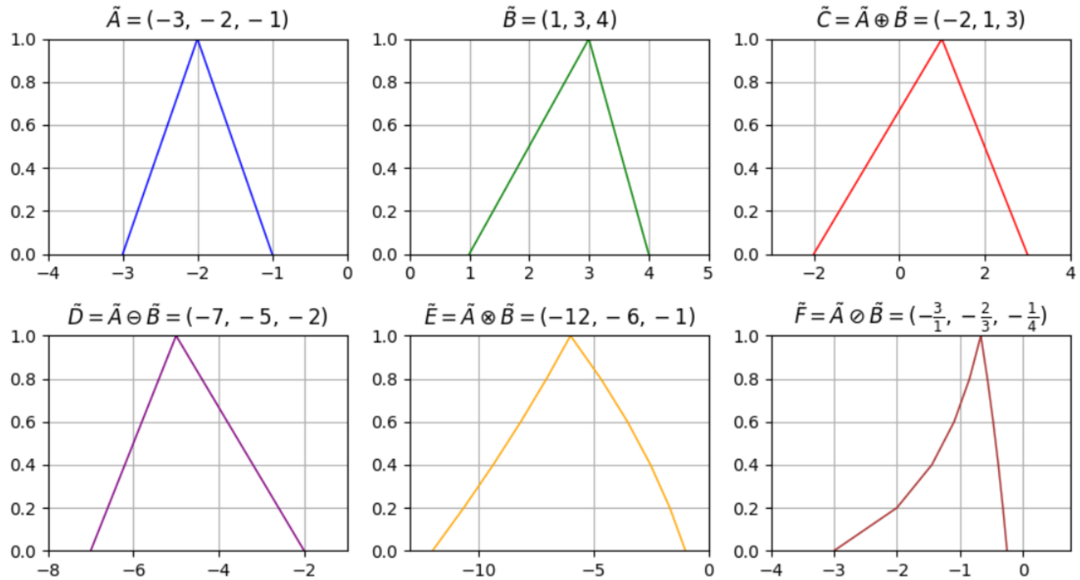

Figure 8: Case: length($\tilde{A}$) < length($\tilde{B}$), considering $\tilde{A}$ is negative and $\tilde{B}$ is positive TFNs.

$\tilde{A}$ and $\tilde{B}$ are around zero TFNs			
$\tilde{A} = (-2,-1,1)$, ${}^{\alpha}\tilde{A} = [\alpha-2, 1-2\alpha]$ and $\tilde{B} = (-2,1,2)$, ${}^{\alpha}\tilde{B} = [3\alpha-2, 2-\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C} = \tilde{A} \oplus \tilde{B}$	$L = [4\alpha-4, 0, \alpha-1, 3-3\alpha]$ $R = [4\alpha-4, \alpha-1, 0, 3-3\alpha]$ ${}^{\alpha}\tilde{C} = [4\alpha-4, 3-3\alpha]$. Using FDT, $\tilde{C} = (-4,0,3)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{B}$ $L = [\alpha-2, 5\alpha-6, 5-6\alpha, 1-2\alpha]$ $R = [5\alpha-6, \alpha-2, 1-2\alpha, 5-6\alpha]$ ${}^{\alpha}\tilde{A} = [\alpha-2, 1-2\alpha]$. Using FDT, $\tilde{A} = (-2,-1,1)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{A}$ $L = [3\alpha-2, 6\alpha-5, 5-4\alpha, 2-\alpha]$ $R = [6\alpha-5, 3\alpha-2, 2-\alpha, 5-4\alpha]$ ${}^{\alpha}\tilde{B} = [3\alpha-2, 2-\alpha]$. Using FDT, $\tilde{B} = (-2,1,2)$.
(ii) $\tilde{D} = \tilde{A} \ominus \tilde{B}$	$L = [-2\alpha, 2\alpha-4, 3-5\alpha, -\alpha-1]$ $R = [2\alpha-4, -\alpha-1, -2\alpha, 3-5\alpha]$ ${}^{\alpha}\tilde{D} = [2\alpha-4, 3-5\alpha]$. Using FDT, $\tilde{D} = (-4,-2,3)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{D} + {}^{\alpha}\tilde{B}$ $L = [5\alpha-6, \alpha-2, 1-2\alpha, 5-6\alpha]$ $R = [5\alpha-6, \alpha-2, 1-2\alpha, 5-6\alpha]$ ${}^{\alpha}\tilde{A} = [\alpha-2, 1-2\alpha]$. Using FDT, $\tilde{A} = (-2,-1,1)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} - {}^{\alpha}\tilde{D}$ $L = [2-\alpha, 6\alpha-5, 5-4\alpha, 3\alpha-2]$ $R = [6\alpha-5, 3\alpha-2, 2-\alpha, 5-4\alpha]$ ${}^{\alpha}\tilde{B} = [3\alpha-2, 2-\alpha]$. Using FDT, $\tilde{B} = (-2,1,2)$.

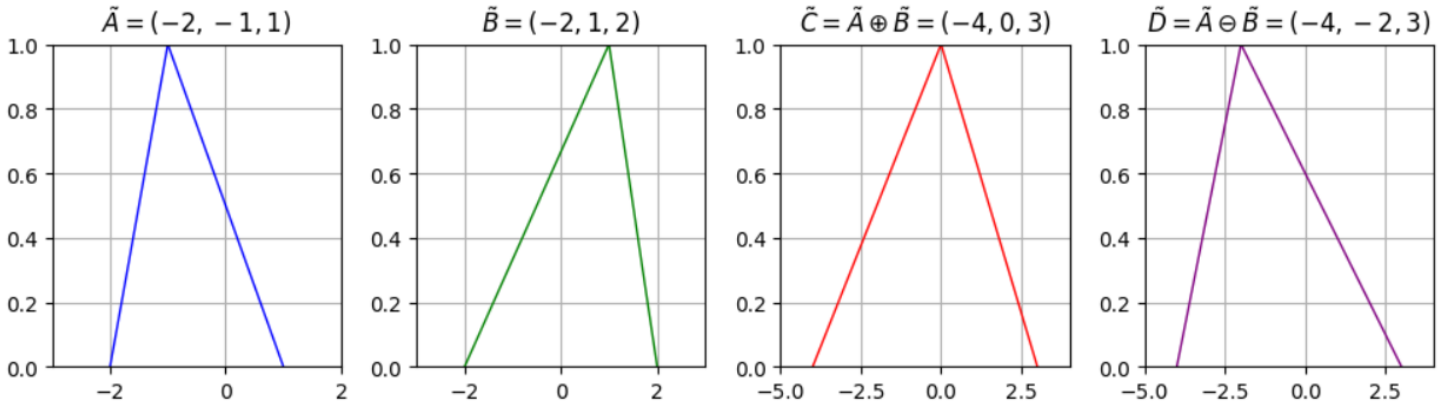

Figure 3: Case: length($\tilde{A}$) < length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are around zero TFNs.

Case 3: length($\tilde{A}$)= length($\tilde{B}$)			
$\tilde{A}$ and $\tilde{B}$ are positive TFNs			
$\tilde{A}=(2,3,5)$, ${}^{\alpha}\tilde{A}=[\alpha+2,5-2\alpha]$ and $\tilde{B}=(4,5,7)$, ${}^{\alpha}\tilde{B}=[\alpha+4,7-2\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C}=\tilde{A}\oplus\tilde{B}$	$L=[2\alpha+6,9-\alpha,9-\alpha,12-4\alpha]$ $R=[2\alpha+6,9-\alpha,9-\alpha,12-4\alpha]$ ${}^{\alpha}\tilde{C}=[2\alpha+6,12-4\alpha]$. Using FDT, $\tilde{C}=(6,8,12)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{B}$ $L=[\alpha+2,4\alpha-1,8-5\alpha,5-2\alpha]$ $R=[4\alpha-1,\alpha+2,5-2\alpha,8-5\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+2,5-2\alpha]$. Using FDT, $\tilde{A}=(2,3,5)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{A}$ $L=[\alpha+4,4\alpha+1,10-5\alpha,7-2\alpha]$ $R=[4\alpha+1,\alpha+4,7-2\alpha,10-5\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha+4,7-2\alpha]$. Using FDT, $\tilde{B}=(4,5,7)$.
(ii) $\tilde{D}=\tilde{A}\ominus\tilde{B}$	$L=[-2,3\alpha-5,1-3\alpha,-2]$ $R=[3\alpha-5,-2,-2,1-3\alpha]$ ${}^{\alpha}\tilde{D}=[3\alpha-5,1-3\alpha]$. Using FDT, $\tilde{D}=(-5,-2,1)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{D}+{}^{\alpha}\tilde{B}$ $L=[4\alpha-1,\alpha+2,5-2\alpha,8-5\alpha]$ $R=[4\alpha-1,\alpha+2,5-2\alpha,8-5\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+2,5-2\alpha]$. Using FDT, $\tilde{A}=(2,3,5)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{A}-{}^{\alpha}\tilde{D}$ $L=[7-2\alpha,4\alpha+1,10-5\alpha,4+\alpha]$ $R=[4\alpha+1,\alpha+4,7-2\alpha,10-5\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha+4,7-2\alpha]$. Using FDT, $\tilde{B}=(4,5,7)$.
(iii) $\tilde{E}=\tilde{A}\otimes\tilde{B}$	$L=[\alpha^2+6\alpha+8,-2\alpha^2+3\alpha+14$ $-2\alpha^2-3\alpha+20,4\alpha^2-24\alpha+35]$ $R=[\alpha^2+6\alpha+8,-2\alpha^2+3\alpha+14$ $-2\alpha^2-3\alpha+20,4\alpha^2-24\alpha+35]$ ${}^{\alpha}\tilde{E}=[\alpha^2+6\alpha+8,4\alpha^2-24\alpha+35]$. Using FDT, $\tilde{E}=(8,15,35)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L=[\alpha^2+6\alpha+8/\alpha+4,$ $\alpha^2+6\alpha+8/7-2\alpha,$ $4\alpha^2-24\alpha+35/\alpha+4,$ $4\alpha^2-24\alpha+35/7-2\alpha]$ $R=[\alpha^2+6\alpha+8/7-2\alpha,$ $\alpha^2+6\alpha+8/\alpha+4,$ $4\alpha^2-24\alpha+35/7-2\alpha,$ $4\alpha^2-24\alpha+35/\alpha+4]$ ${}^{\alpha}\tilde{A}=[\alpha+2,5-2\alpha]$. Using FDT, $\tilde{A}=(2,3,5)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L=[\alpha^2+6\alpha+8/\alpha+2,$ $\alpha^2+6\alpha+8/5-2\alpha,$ $4\alpha^2-24\alpha+35/\alpha+2,$ $4\alpha^2-24\alpha+35/5-2\alpha]$ $R=[\alpha^2+6\alpha+8/5-2\alpha,$ $\alpha^2+6\alpha+8/\alpha+2,$ $4\alpha^2-24\alpha+35/5-2\alpha,$ $4\alpha^2-24\alpha+35/\alpha+2]$ ${}^{\alpha}\tilde{B}=[\alpha+4,7-2\alpha]$. Using FDT, $\tilde{B}=(4,5,7)$.

Arithmetic operations	Calculation	Getting Operands	
		Calculation for ${}^\alpha\tilde{A} = {}^\alpha\tilde{F} * {}^\alpha\tilde{B}$	Calculation for ${}^\alpha\tilde{B} = {}^\alpha\tilde{A}/{}^\alpha\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [\alpha + 2/\alpha + 4, \alpha + 2/7 - 2\alpha, 5 - 2\alpha/\alpha + 4, 5 - 2\alpha/7 - 2\alpha]$ $R = [\alpha + 2/7 - 2\alpha, \alpha + 2/\alpha + 4, 5 - 2\alpha/7 - 2\alpha, 5 - 2\alpha/\alpha + 4]$ ${}^\alpha\tilde{F} = [\alpha + 2/7 - 2\alpha, 5 - 2\alpha/\alpha + 4]$. Using FDT, $\tilde{F} = (2/7, 3/5, 5/4)$.	$L = [\alpha^2 + 6\alpha + 8/7 - 2\alpha, \alpha + 2, 5 - 2\alpha, 4\alpha^2 - 24\alpha + 35/\alpha + 4]$ $R = [\alpha^2 + 6\alpha + 8/7 - 2\alpha, \alpha + 2, 5 - 2\alpha, 4\alpha^2 - 24\alpha + 35/\alpha + 4]$ ${}^\alpha\tilde{A} = [\alpha + 2, 5 - 2\alpha]$. Using FDT, $\tilde{A} = (2, 3, 5)$.	$L = [7 - 2\alpha, \alpha^2 + 6\alpha + 8/5 - 2\alpha, 4\alpha^2 - 24\alpha + 35/\alpha + 2, \alpha + 4]$ $R = [\alpha^2 + 6\alpha + 8/5 - 2\alpha, \alpha + 4, 7 - 2\alpha, 4\alpha^2 - 24\alpha + 35/\alpha + 2]$ ${}^\alpha\tilde{B} = [\alpha + 4, 7 - 2\alpha]$. Using FDT, $\tilde{B} = (4, 5, 7)$.

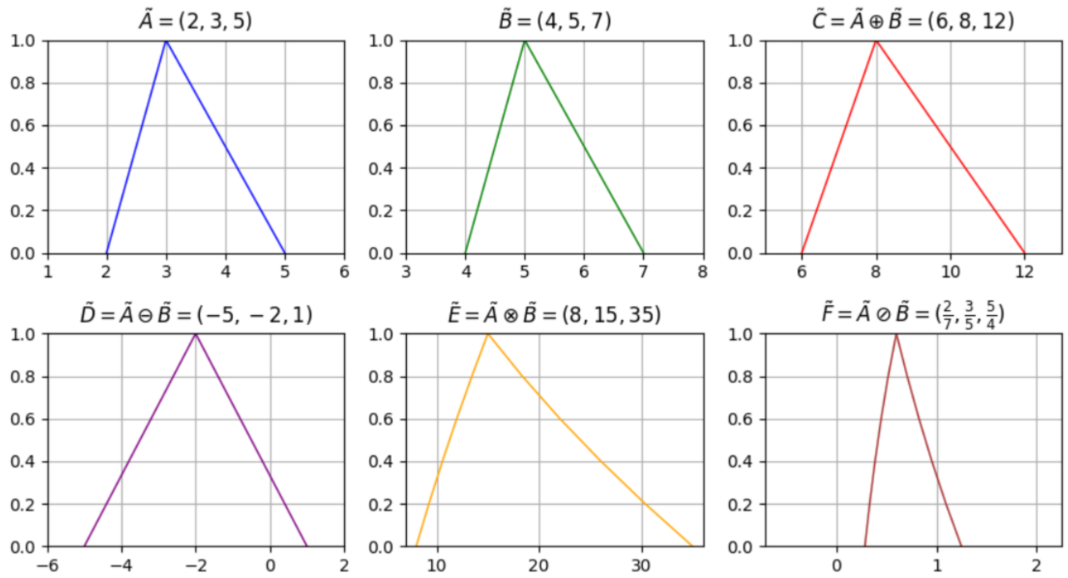

Figure 10: Case: length($\tilde{A}$) = length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are positive TFNs.

$\tilde{A}$ and $\tilde{B}$ are negative TFNs			
$\tilde{A}=(-4,-3,-2)$, ${}^{\alpha}\tilde{A}=[\alpha-4,-2-\alpha]$ and $\tilde{B}=(-5,-4,-3)$, ${}^{\alpha}\tilde{B}=[\alpha-5,-3-\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C}=\tilde{A}\oplus\tilde{B}$	$L=[2\alpha-9,-7,-7,-5-2\alpha]$ $R=[2\alpha-9,-7,-7,-5-2\alpha]$ ${}^{\alpha}\tilde{C}=[2\alpha-9,-5-2\alpha]$. Using FDT, $\tilde{C}=(-9,-7,-5)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{B}$ $L=[3\alpha-6,\alpha-4,-2-\alpha,-3\alpha]$ $R=[3\alpha-6,\alpha-4,-2-\alpha,-3\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha-4,-2-\alpha]$. Using FDT, $\tilde{A}=(-4,-3,-2)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{A}$ $L=[\alpha-5,3\alpha-7,-1-3\alpha,-3-\alpha]$ $R=[3\alpha-7,\alpha-5,-3-\alpha,-1-3\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-5,-3-\alpha]$. Using FDT, $\tilde{B}=(-5,-4,-3)$.
(ii) $\tilde{D}=\tilde{A}\ominus\tilde{B}$	$L=[1,2\alpha-1,3-2\alpha,1]$ $R=[2\alpha-1,1,1,3-2\alpha]$ ${}^{\alpha}\tilde{D}=[2\alpha-1,3-2\alpha]$. Using FDT, $\tilde{D}=(-1,1,3)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{D}+{}^{\alpha}\tilde{B}$ $L=[3\alpha-6,\alpha-4,-2-\alpha,-3\alpha]$ $R=[3\alpha-6,\alpha-4,-2-\alpha,-3\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha-4,-2-\alpha]$. Using FDT, $\tilde{A}=(-4,-3,-2)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{A}-{}^{\alpha}\tilde{D}$ $L=[-\alpha-3,3\alpha-7,-1-3\alpha,\alpha-5]$ $R=[3\alpha-7,\alpha-5,-\alpha-3,-1-3\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-5,-\alpha-3]$. Using FDT, $\tilde{B}=(-5,-4,-3)$.
(iii) $\tilde{E}=\tilde{A}\otimes\tilde{B}$	$L=[\alpha^2-9\alpha+20,-\alpha^2+\alpha+12$ $-\alpha^2+3\alpha+10,\alpha^2+5\alpha+6]$ $R=[\alpha^2+5\alpha+6,-\alpha^2+3\alpha+10,$ $-\alpha^2+\alpha+12,\alpha^2-9\alpha+20]$ ${}^{\alpha}\tilde{E}=[\alpha^2+5\alpha+6,\alpha^2-9\alpha+20]$. Using FDT, $\tilde{E}=(6,12,20)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L=[\alpha^2+5\alpha+6/\alpha-5,$ $\alpha^2+5\alpha+6/-3-\alpha,$ $\alpha^2-9\alpha+20/\alpha-5,$ $\alpha^2-9\alpha+20/-3-\alpha]$ $R=[\alpha^2-9\alpha+20/-3-\alpha,$ $\alpha^2-9\alpha+20/\alpha-5,$ $\alpha^2+5\alpha+6/-3-\alpha,$ $\alpha^2+5\alpha+6/\alpha-5]$ ${}^{\alpha}\tilde{A}=[\alpha-4,-2-\alpha]$. Using FDT, $\tilde{A}=(-4,-3,-2)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L=[\alpha^2+5\alpha+6/\alpha-4,$ $\alpha^2+5\alpha+6/-2-\alpha,$ $\alpha^2-9\alpha+20/\alpha-4,$ $\alpha^2-9\alpha+20/-2-\alpha]$ $R=[\alpha^2-9\alpha+20/-2-\alpha,$ $\alpha^2-9\alpha+20/\alpha-4,$ $\alpha^2+5\alpha+6/-2-\alpha,$ $\alpha^2+5\alpha+6/\alpha-4]$ ${}^{\alpha}\tilde{B}=[\alpha-5,-3-\alpha]$. Using FDT, $\tilde{B}=(-5,-4,-3)$.

Arithmetic operations	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A}/{}^{\alpha}\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [\alpha - 4/\alpha - 5, \alpha - 4/ - 3 - \alpha, - 2 - \alpha/\alpha - 5, -2 - \alpha/ - 3 - \alpha]$ $R = [-2 - \alpha/\alpha - 5, -2 - \alpha/ - 3 - \alpha, \alpha - 4/\alpha - 5, \alpha - 4/ - 3 - \alpha]$ ${}^{\alpha}\tilde{F} = [-2 - \alpha/\alpha - 5, \alpha - 4/ - 3 - \alpha]$. Using FDT, $\tilde{F} = (2/5, 3/4, 4/3)$.	$L = [-2 - \alpha, \alpha^2 + 5\alpha + 6/\alpha - 5, \alpha^2 - 9\alpha + 20/ - 3 - \alpha, \alpha - 4]$ $R = [\alpha^2 - 9\alpha + 20/ - 3 - \alpha, \alpha - 4, - 2 - \alpha, \alpha^2 + 5\alpha + 6/\alpha - 5]$ ${}^{\alpha}\tilde{A} = [\alpha - 4, -2 - \alpha]$. Using FDT, $\tilde{A} = (-4, -3, -2)$.	$L = [\alpha^2 - 9\alpha + 20/ - 2 - \alpha, -3 - \alpha, \alpha - 5, \alpha^2 + 5\alpha + 6/\alpha - 4]$ $R = [\alpha^2 - 9\alpha + 20/ - 2 - \alpha, \alpha - 5, - 3 - \alpha, \alpha^2 + 5\alpha + 6/\alpha - 4]$ ${}^{\alpha}\tilde{B} = [\alpha - 5, -3 - \alpha]$. Using FDT, $\tilde{B} = (-5, -4, -3)$.

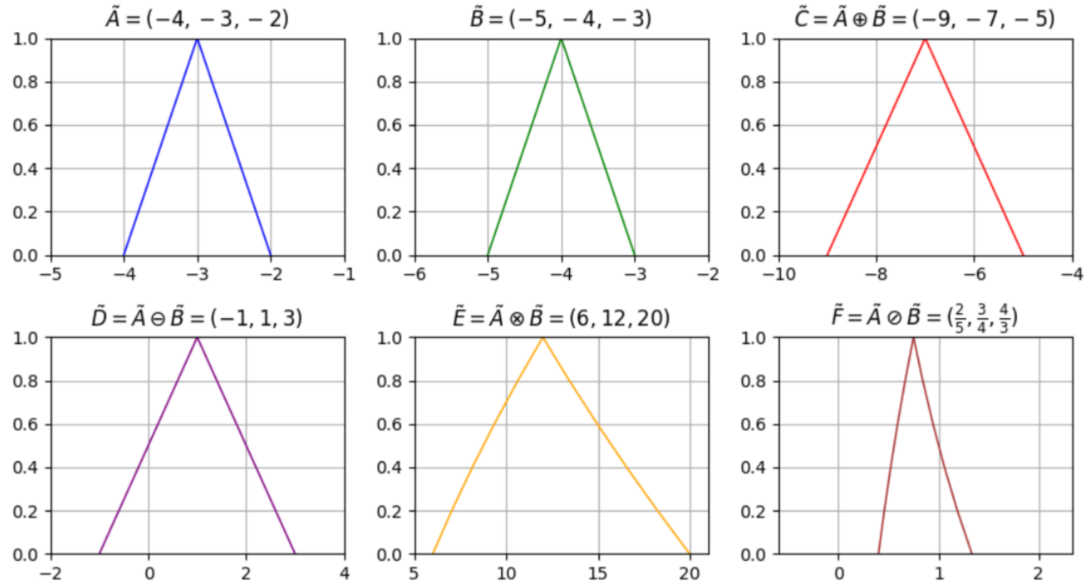

Figure 11: Case: length($\tilde{A}$) = length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are negative TFNs.

$\tilde{A}$ is negative and $\tilde{B}$ is positive TFNs			
$\tilde{A}=(2,3,4)$, ${}^{\alpha}\tilde{A}=[\alpha+2,4-\alpha]$ and $\tilde{B}=(-3,-2,-1)$, ${}^{\alpha}\tilde{B}=[\alpha-3,-1-\alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C}=\tilde{A}\oplus\tilde{B}$	$L=[2\alpha-1,1,1,3-2\alpha]$ $R=[2\alpha-1,1,1,3-2\alpha]$ ${}^{\alpha}\tilde{C}=[2\alpha-1,3-2\alpha]$. Using FDT, $\tilde{C}=(-1,1,3)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{B}$ $L=[\alpha+2,3\alpha,6-3\alpha,4-\alpha]$ $R=[3\alpha,\alpha+2,4-\alpha,6-3\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+2,4-\alpha]$. Using FDT, $\tilde{A}=(2,3,4)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{C}-{}^{\alpha}\tilde{A}$ $L=[\alpha-3,3\alpha-5,1-3\alpha,-1-\alpha]$ $R=[3\alpha-5,\alpha-3,-1-\alpha,1-3\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-3,-1-\alpha]$. Using FDT, $\tilde{B}=(-3,-2,-1)$.
(ii) $\tilde{D}=\tilde{A}\ominus\tilde{B}$	$L=[5,2\alpha+3,7-2\alpha,5]$ $R=[2\alpha+3,5,5,7-2\alpha]$ ${}^{\alpha}\tilde{D}=[2\alpha+3,7-2\alpha]$. Using FDT, $\tilde{D}=(3,5,7)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{D}+{}^{\alpha}\tilde{B}$ $L=[3\alpha,\alpha+2,4-\alpha,6-3\alpha]$ $R=[3\alpha,\alpha+2,4-\alpha,6-3\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+2,4-\alpha]$. Using FDT, $\tilde{A}=(2.3.4)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{A}-{}^{\alpha}\tilde{D}$ $L=[-1-\alpha,3\alpha-5,1-3\alpha,\alpha-3]$ $R=[3\alpha-5,\alpha-3,-1-\alpha,1-3\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-3,-1-\alpha]$. Using FDT, $\tilde{B}=(-3,-2,-1)$.
(iii) $\tilde{E}=\tilde{A}\otimes\tilde{B}$	$L=[\alpha^2-\alpha-6,-\alpha^2-3\alpha-2,$ $-\alpha^2+7\alpha-12,\alpha^2-3\alpha-4]$ $R=[-\alpha^2+7\alpha-12,\alpha^2-\alpha-6,$ $\alpha^2-3\alpha-4,-\alpha^2-3\alpha-2]$ ${}^{\alpha}\tilde{E}=[-\alpha^2+7\alpha-12,-\alpha^2-3\alpha-2]$. Using FDT, $\tilde{E}=(-12,-6,-2)$.	Calculation for ${}^{\alpha}\tilde{A}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{B}$ $L=[-\alpha^2+7\alpha-12/\alpha-3,$ $-\alpha^2+7\alpha-12/-1-\alpha,$ $-\alpha^2-3\alpha-2/\alpha-3,$ $-\alpha^2-3\alpha-2/-1-\alpha]$ $R=[-\alpha^2-3\alpha-2/\alpha-3,$ $-\alpha^2-3\alpha-2/-1-\alpha,$ $-\alpha^2+7\alpha-12/\alpha-3,$ $-\alpha^2+7\alpha-12/-1-\alpha]$ ${}^{\alpha}\tilde{A}=[\alpha+2,4-\alpha]$. Using FDT, $\tilde{A}=(2,3,4)$.	Calculation for ${}^{\alpha}\tilde{B}={}^{\alpha}\tilde{E}/{}^{\alpha}\tilde{A}$ $L=[-\alpha^2+7\alpha-12/\alpha+2,$ $-\alpha^2+7\alpha-12/4-\alpha,$ $-\alpha^2-3\alpha-2/\alpha+2,$ $-\alpha^2-3\alpha-2/4-\alpha]$ $R=[-\alpha^2+7\alpha-12/\alpha+2,$ $-\alpha^2+7\alpha-12/4-\alpha,$ $-\alpha^2-3\alpha-2/\alpha+2,$ $-\alpha^2-3\alpha-2/4-\alpha]$ ${}^{\alpha}\tilde{B}=[\alpha-3,-1-\alpha]$. Using FDT, $\tilde{B}=(-3,-2,-1)$.

Arithmetic operations	Calculation	Getting Operands	
		Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{F} * {}^{\alpha}\tilde{B}$	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A}/{}^{\alpha}\tilde{F}$
(iv) $\tilde{F} = \tilde{A} \oslash \tilde{B}$	$L = [\alpha + 2/\alpha - 3, \alpha + 2/ - 1 - \alpha, 4 - \alpha/\alpha - 3, 4 - \alpha/ - 1 - \alpha]$ $R = [4 - \alpha/ - 1 - \alpha, \alpha + 2/ - 1 - \alpha, 4 - \alpha/\alpha - 3, \alpha + 2/\alpha - 3]$ ${}^{\alpha}\tilde{F} = [4 - \alpha/ - 1 - \alpha, \alpha + 2/\alpha - 3]$. Using FDT, $\tilde{F} = (-4, -3/2, -2/3)$.	$L = [-\alpha^2 + 7\alpha - 12/ - 1 - \alpha, 4 - \alpha, \alpha + 2, -\alpha^2 - 3\alpha - 2/\alpha - 3]$ $R = [-\alpha^2 - 3\alpha - 2/\alpha - 3, \alpha + 2, 4 - \alpha, -\alpha^2 + 7\alpha - 12/ - 1 - \alpha]$ ${}^{\alpha}\tilde{A} = [\alpha + 2, 4 - \alpha]$. Using FDT, $\tilde{A} = (2, 3, 4)$.	$L = [-\alpha^2 - 3\alpha - 2/4 - \alpha, \alpha - 3, - 1 - \alpha, -\alpha^2 + 7\alpha - 12/\alpha + 2]$ $R = [-\alpha^2 + 7\alpha - 12/\alpha + 2, \alpha - 3, - 1 - \alpha, -\alpha^2 - 3\alpha - 2/4 - \alpha]$ ${}^{\alpha}\tilde{B} = [\alpha - 3, -1 - \alpha]$. Using FDT, $\tilde{B} = (-3, -2, -1)$.

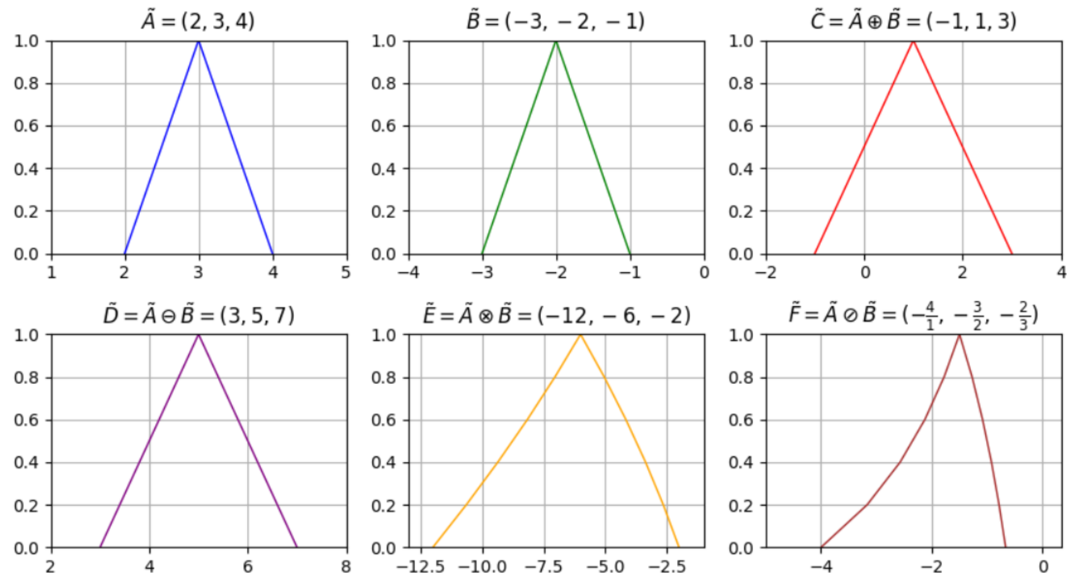

Figure 12: Case: length($\tilde{A}$) = length($\tilde{B}$), considering $\tilde{A}$ is positive and $\tilde{B}$ is negative TFNs.

$\tilde{A}$ and $\tilde{B}$ are around zero TFNs			
$\tilde{A} = (-3, -2, 1)$, ${}^{\alpha}\tilde{A} = [\alpha - 3, 1 - 3\alpha]$ and $\tilde{B} = (-1, 2, 3)$, ${}^{\alpha}\tilde{B} = [3\alpha - 1, 3 - \alpha]$			
Arithmetic operations	**Calculation**	**Getting Operands**	
(i) $\tilde{C} = \tilde{A} \oplus \tilde{B}$	$L = [4\alpha - 4, 0, 0, 4 - 4\alpha]$ $R = [4\alpha - 4, 0, 0, 4 - 4\alpha]$ ${}^{\alpha}\tilde{C} = [4\alpha - 4, 4 - 4\alpha]$. Using FDT, $\tilde{C} = (-4, 0, 4)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{B}$ $L = [\alpha - 3, 5\alpha - 7, 5 - 7\alpha, 1 - 3\alpha]$ $R = [5\alpha - 7, \alpha - 3, 1 - 3\alpha, 5 - 7\alpha]$ ${}^{\alpha}\tilde{A} = [\alpha - 3, 1 - 3\alpha]$. Using FDT, $\tilde{A} = (-3, -2, 1)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{C} - {}^{\alpha}\tilde{A}$ $L = [3\alpha - 1, 7\alpha - 3, 7 - 5\alpha, 3 - \alpha]$ $R = [7\alpha - 3, 3\alpha - 1, 3 - \alpha, 7 - 5\alpha]$ ${}^{\alpha}\tilde{B} = [3\alpha - 1, 3 - \alpha]$. Using FDT, $\tilde{B} = (-1, 2, 3)$.
(ii) $\tilde{D} = \tilde{A} \ominus \tilde{B}$	$L = [-2\alpha - 2, 2\alpha - 6, 2 - 6\alpha, -2 - 2\alpha]$ $R = [2\alpha - 6, -2 - 2\alpha, -2\alpha - 2, 2 - 6\alpha]$ ${}^{\alpha}\tilde{D} = [2\alpha - 6, 2 - 6\alpha]$. Using FDT, $\tilde{D} = (-6, -4, 2)$.	Calculation for ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{D} + {}^{\alpha}\tilde{B}$ $L = [5\alpha - 7, \alpha - 3, 1 - 3\alpha, 5 - 7\alpha]$ $R = [5\alpha - 7, \alpha - 3, 1 - 3\alpha, 5 - 7\alpha]$ ${}^{\alpha}\tilde{A} = [\alpha - 3, 1 - 3\alpha]$. Using FDT, $\tilde{A} = (-3, -2, 1)$.	Calculation for ${}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{A} - {}^{\alpha}\tilde{D}$ $L = [3 - \alpha, 7\alpha - 5, 7 - 5\alpha, 3\alpha - 1]$ $R = [7\alpha - 5, 3\alpha - 1, 3 - \alpha, 7 - 5\alpha]$ ${}^{\alpha}\tilde{B} = [3\alpha - 1, 3 - \alpha]$. Using FDT, $\tilde{B} = (-1, 2, 3)$.

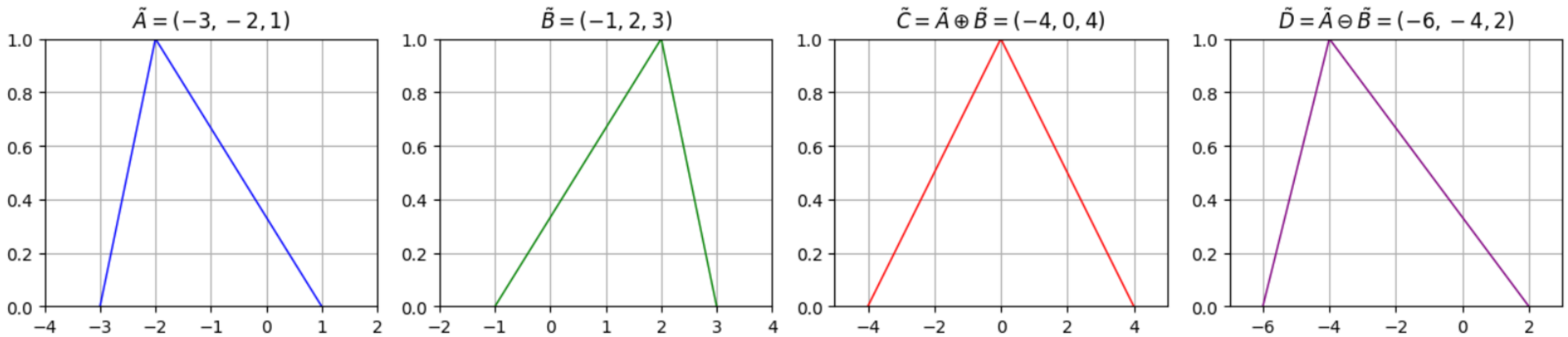

Figure 13: Case: length($\tilde{A}$) = length($\tilde{B}$), considering $\tilde{A}$ and $\tilde{B}$ are around zero TFNs.

6 Fuzzy Scalar Equations

In this section, we make use of the our newly proposed fuzzy arithmetic operations to solve scalar fuzzy equations of the form $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \otimes \tilde{X} = \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are given fuzzy numbers, and $\tilde{X}$ is the unknown fuzzy number to be determined. The solution and verification steps are both calculated using the proposed fuzzy arithmetic approach.

To solve each equation, we express $\tilde{X}$ in terms of known fuzzy numbers as $\tilde{X} = \tilde{B} \ominus \tilde{A}$ and $\tilde{X} = \tilde{B} \oslash \tilde{A}$. Find the value of the expression on the right and assign it to $\tilde{X}$ as per computations described above in section(3). We also verify the value obtained for the unknown by putting it in the given equations and comparing both the sides.

Let $\tilde{A}$ and $\tilde{B}$ be the operands in the equations $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \otimes \tilde{X} = \tilde{B}$, where $\tilde{X}$ denotes the fuzzy solution we are seeking. For $\alpha \in (0, 1]$, let the $\alpha-$cut for $\tilde{A}$, $\tilde{B}$ and $\tilde{X}$ be ${}^{\alpha}\tilde{A} = [\underline{A}, \overline{A}]$, ${}^{\alpha}\tilde{B} = [\underline{B}, \overline{B}]$ and ${}^{\alpha}\tilde{X} = [\underline{X}, \overline{X}]$. Here, ${}^{\alpha}\tilde{X}$ is obtained by the newly proposed technique. Then, the fuzzy equation has a solution in fuzzy form if and only if the following two conditions are satisfied:

(i) For every $\alpha \in (0, 1]$, ${}^{\alpha}\underline{X} \leq {}^{\alpha}\overline{X}$

(ii) If $\alpha \leq \beta$, then the interval bounds satisfy the following relation:
${}^{\alpha}\underline{X} \leq {}^{\beta}\underline{X} \leq {}^{\beta}\overline{X} \leq {}^{\alpha}\overline{X}$.

Property (i) makes sure that the fuzzy numbers $\tilde{X}$ has a proper support and Property (ii) ensures that the $\alpha-$ cuts are proper nested to qualify as a fuzzy number.
If a solution ${}^{\alpha}\tilde{X}$ exists for every $\alpha \in (0, 1]$ and property (ii) is satisfied, then the solution $\tilde{X}$ of the fuzzy equation is constructed by the First Decomposition Theorem as in (2.8).

Remark 6.1 *Generally, to obtain the unknown $\tilde{X}$ we choose r_2 and r_3 from R. However, in some cases, selecting these elements does not yield valid fuzzy solutions. In such situations, we consider the outer elements (i.e., the r_1 and r_4 elements), which leads to a fuzzy solution. However, a limitation is that selecting outer elements often results in larger support.*

Remark 6.2 *When we verify the solution obtained using our method by putting it in L.H.S. to get R.H.S., we observe that the R.H.S. in few cases is obtained by considering r_2 and r_3 instead of r_1 and r_4 of R. Thus, the R.H.S. may by obtained either by r_1 and r_4 orr_2 and r_3 elements of R. Hence, we succeed using this method.*

6.1 Illustrative Examples :

1. Equation : $\tilde{A} \oplus \tilde{X} = \tilde{B}$

Example 1. Consider $\tilde{A} = (1, 3, 4)$ and $\tilde{B} = (-2, 1, 3)$. The $\alpha-$cut of $\tilde{A}$ is ${}^{\alpha}\tilde{A} = [2\alpha + 1, 4 - \alpha]$ and $\alpha-$cut of $\tilde{B}$ is ${}^{\alpha}\tilde{B} = [3\alpha - 2, 3 - 2\alpha]$.
To solve given equation for $\tilde{X}$, as

$$ {}^{\alpha}\tilde{X} = {}^{\alpha}\tilde{B} - {}^{\alpha}\tilde{A}. $$

Let us compute L for ${}^{\alpha}\tilde{B} - {}^{\alpha}\tilde{A}$ as,

$$ L = [\alpha - 3, 4\alpha - 6, 2 - 4\alpha, -\alpha - 1] $$

Applying the $fsort()$ function to L as,

$$ R = fsort(L) = [4\alpha - 6,\ \alpha - 3,\ -\alpha - 1,\ 2 - 4\alpha] $$

Therefore, ${}^{\alpha}\tilde{X} = [4\alpha - 6, 2 - 4\alpha]$. Using Theorem (2.8), $\tilde{X} = (-6,\ -2,\ 2)$.
Verification: Starting from the Left-hand side(L.H.S.) of the equation, we have

$$\begin{aligned}
{}^{\alpha}\tilde{A} + {}^{\alpha}\tilde{X} &= [2\alpha + 1, 4 - \alpha] + [4\alpha - 6, 2 - 4\alpha] \\
L &= [6\alpha - 5,\ 3 - 2\alpha,\ 3\alpha - 2,\ 6 - 5\alpha] \\
R &= [6\alpha - 5,\ 3\alpha - 2,\ 3 - 2\alpha,\ 6 - 5\alpha]
\end{aligned}$$

From R considering the first and last elements, we get $[6\alpha - 5,\ 6 - 5\alpha]$. Using Theorem (2.8), we get $(-5, 1, 6)$, which is same as $\tilde{B}$ at core. Hence, we conclude that the obtained solution is correct, as shown in Figure (3).

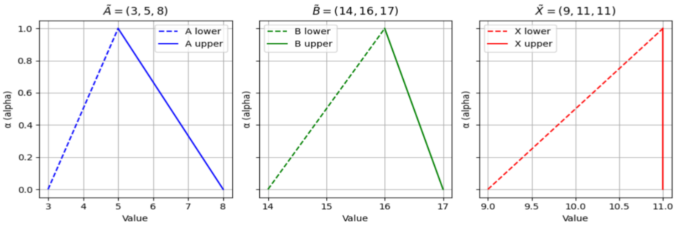

Figure 3: α-cut Plots of $\tilde{A}$, $\tilde{B}$, and Solution $\tilde{X}$.

2. Equation : $\tilde{A} \otimes \tilde{X} = \tilde{B}$

Example 1: Consider $\tilde{A} = (2, 5, 7)$ and $\tilde{B} = (3, 4, 6)$. The $\alpha-$cut of $\tilde{A}$ is ${}^{\alpha}\tilde{A} = [3\alpha + 2, 7 - 2\alpha]$ and $\alpha-$cut of $\tilde{B}$ is ${}^{\alpha}\tilde{B} = [\alpha + 3, 6 - 2\alpha]$. To solve given equation for $\tilde{X}$, we take $\alpha-$cut on both the sides, we get

$${}^{\alpha}\tilde{X} = \frac{{}^{\alpha}\tilde{B}}{{}^{\alpha}\tilde{A}}.$$

We compute L for $\frac{{}^{\alpha}\tilde{B}}{{}^{\alpha}\tilde{A}}$ as,

$$L = \left[\frac{\alpha + 3}{3\alpha + 2}, \frac{\alpha + 3}{7 - 2\alpha}, \frac{6 - 2\alpha}{3\alpha + 2}, \frac{6 - 2\alpha}{7 - 2\alpha}\right]$$

Then find R as,

$$R = fsort(L) = \left[\frac{\alpha + 3}{7 - 2\alpha}, \frac{6 - 2\alpha}{7 - 2\alpha}, \frac{\alpha + 3}{3\alpha + 2}, \frac{6 - 2\alpha}{3\alpha + 2}\right]$$

Therefore, ${}^{\alpha}\tilde{X} = \left[\frac{\alpha+3}{7-2\alpha}, \frac{6-2\alpha}{3\alpha+2}\right]$. Using Theorem (2.8), $\tilde{X} = (0.4285, 0.8, 3)$.
Verification: Let us Start from the Left-hand side(L.H.S.) of the equation, we have

$$\begin{aligned}
{}^{\alpha}\tilde{A} * {}^{\alpha}\tilde{X} &= [3\alpha + 2, 7 - 2\alpha] * \left[\frac{\alpha + 3}{7 - 2\alpha}, \frac{6 - 2\alpha}{3\alpha + 2}\right] \\
L &= \left[\frac{3\alpha^2 + 11\alpha + 6}{7 - 2\alpha}, \alpha + 3, 6 - 2\alpha, \frac{2\alpha^2 - 13\alpha + 21}{3\alpha + 2}\right] \\
R &= \left[\frac{3\alpha^2 + 11\alpha + 6}{7 - 2\alpha}, \alpha + 3, 6 - 2\alpha, \frac{2\alpha^2 - 13\alpha + 21}{3\alpha + 2}\right]
\end{aligned}$$

From R considering the first and last elements, we get $\left[\frac{3\alpha^2+11\alpha+6}{7-2\alpha}, \frac{2\alpha^2-13\alpha+21}{3\alpha+2}\right]$. Using Theorem (2.8 we obtain the resultant fuzzy triangular number $(0.8571, 4, 10.5)$. Comparing its core with the core of $\tilde{B}$, we observe that which is same as $\tilde{B}$ at core, as shown in Figure (4). Hence, we conclude that the obtained solution is correct.

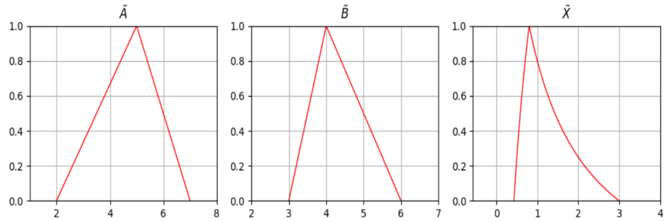

Figure 4: α-cut plots of $\tilde{A}$, $\tilde{B}$, and solution $\tilde{X}$.

Example 2: [3] Consider $\tilde{A} = (2, 4, 6)$ and $\tilde{B} = (6, 8, 10)$. The $\alpha-$cut of $\tilde{A}$ is ${}^{\alpha}\tilde{A} = [2\alpha + 2, 6 - 2\alpha]$ and $\alpha-$cut of $\tilde{B}$ is ${}^{\alpha}\tilde{B} = [2\alpha + 6, 10 - 2\alpha]$.
To solve given equation for $\tilde{X}$, we take $\alpha-$cut on both the sides, we get

$${}^{\alpha}\tilde{X} = \frac{{}^{\alpha}\tilde{B}}{{}^{\alpha}\tilde{A}}.$$

We compute L for $\frac{{}^{\alpha}\tilde{B}}{{}^{\alpha}\tilde{A}}$ as,

$$L = \left[\frac{2\alpha + 6}{2\alpha + 2}, \frac{2\alpha + 6}{6 - 2\alpha}, \frac{10 - 2\alpha}{2\alpha + 2}, \frac{10 - 2\alpha}{6 - 2\alpha}\right]$$

Then find R as,

$$R = fsort(L) = \left[\frac{2\alpha + 6}{6 - 2\alpha}, \frac{10 - 2\alpha}{6 - 2\alpha}, \frac{2\alpha + 6}{2\alpha + 2}, \frac{10 - 2\alpha}{2\alpha + 2}\right]$$

Therefore, ${}^{\alpha}\tilde{X} = \left[\frac{2\alpha+6}{6-2\alpha}, \frac{10-2\alpha}{2\alpha+2}\right]$. Using Theorem (2.8), $\tilde{X} = (1, 2, 5)$.
Verification: Let us Start from the Left-hand side(L.H.S.) of the equation, we have

$${}^{\alpha}\tilde{A} * {}^{\alpha}\tilde{X} = [2\alpha + 2, 6 - 2\alpha] * \left[\frac{2\alpha + 6}{6 - 2\alpha}, \frac{10 - 2\alpha}{2\alpha + 2}\right]$$

$$L = \left[\frac{2\alpha^2 + 8\alpha + 6}{3 - \alpha}, \ 10 - 2\alpha, \ 2\alpha + 6, \ \frac{2\alpha^2 - 16\alpha + 30}{\alpha + 1}\right]$$

$$R = \left[\frac{2\alpha^2 + 8\alpha + 6}{3 - \alpha}, \ 2\alpha + 6, 10 - 2\alpha, \ \frac{2\alpha^2 - 16\alpha + 30}{\alpha + 1}\right]$$

Considering the first and last elements from R, we get $\left[\frac{2\alpha^2+8\alpha+6}{3-\alpha}, \frac{2\alpha^2-16\alpha+30}{\alpha+1}\right]$. Using Theorem (2.8), we obtain the resultant fuzzy triangular number $(2, 8, 30)$. Comparing its core with the core of $\tilde{B}$, we observe that which is same as $\tilde{B}$ at core, as shown in Figure (5). Hence, we conclude that the obtained solution is correct.

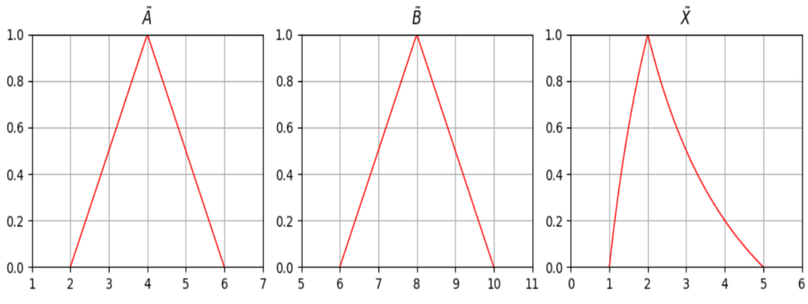

Figure 5: α-cut Plots of $\tilde{A}$, $\tilde{B}$, and Solution $\tilde{X}$.

Remark: Using the standard interval method [3], the solution to this equation does not exist. However, with the proposed method, a solution does exist.

7 Fuzzy System of Linear Equations

A general fuzzy system of n equations with n unknowns can be expressed as, where $n \geq 2$:

$$\begin{aligned} \tilde{a}_{11} \otimes \tilde{x}_1 \oplus \tilde{a}_{12} \otimes \tilde{x}_2 \oplus \cdots \oplus \tilde{a}_{1n} \otimes \tilde{x}_n &= \tilde{b}_1 \\ \tilde{a}_{21} \otimes \tilde{x}_1 \oplus \tilde{a}_{22} \otimes \tilde{x}_2 \oplus \cdots \oplus \tilde{a}_{2n} \otimes \tilde{x}_n &= \tilde{b}_2 \\ \vdots \qquad \qquad \vdots \quad \ \vdots \\ \tilde{a}_{n1} \otimes \tilde{x}_1 \oplus \tilde{a}_{n2} \otimes \tilde{x}_2 \oplus \cdots \oplus \tilde{a}_{nn} \otimes \tilde{x}_n &= \tilde{b}_n \end{aligned} \tag{2}$$

In compact matrix form, this system is written as:

$$\tilde{A} \otimes \tilde{X} = \tilde{B} \tag{3}$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix}.$$

The solution of (3) is obtained by Cramer's rule similar as in classical set up.

7.1 Fuzzy Cramer's Rule

In classical case, Cramer's Rule can be applied for solving a system of n linear equations with n unknowns, provided that the coefficient matrix is square and has a non-zero determinant. In this study, we extended this method, when both the coefficients and constants on R.H.S. are represented as fuzzy numbers to capture imprecision in the system.

To solve for $\tilde{x}_i$, the modified matrix $\tilde{A}_i$ is constructed by replacing the i^{th} column of $\tilde{A}$ with the fuzzy constant vector $\tilde{B}$,

$$\tilde{A}_i = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{b}_1 & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \cdots & \tilde{b}_2 & \cdots & \tilde{a}_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{b}_n & \cdots & \tilde{a}_{nn} \end{bmatrix}.$$

The fuzzy solution is then obtained using determinants:

$$\tilde{x}_i = \frac{\det(\tilde{A}_i)}{\det(\tilde{A})}, \qquad i = 1, 2, \ldots, n.$$

Here, $\det(\tilde{A}) = D$ denotes the determinant of the fuzzy coefficient matrix, and $\det(\tilde{A}_i) = D_i$ denotes the determinant of the modified matrix. To evaluate D and D_i, fuzzy arithmetic operations such as addition, subtraction, multiplication, and division are applied consistently within the fuzzy framework. In higher-order calculations, repeated multiplications increases the powers of α, which makes the calculations more complicated. To keep the process simple, after each step we convert the result of α-cut back into a triangular fuzzy number. This means that the imprecision present in the coefficients and constants is carried forward into the solutions in a natural way. The fuzzy version of Cramer's Rule gives analytical solutions and presents them in a straightforward and understandable way. This makes it a useful tool both for checking theoretical results and for making decisions in situations involving imprecise data.

We apply the proposed method for the application of supply and demand case analyzed by [13].

7.2 Application to Fuzzy Supply–Demand Model

The supply-and-demand model is a fundamental concept in economics that explains how the interaction of buyers and sellers determines the price and quantity of goods in a market. Traditionally, supply and demand functions are expressed in terms of price alone; however, in real markets, additional factors such as consumer income, production costs, substitutes, and expectations influence the outcome. These factors introduce imprecision, which can be effectively represented using fuzzy parameters. By formulating the supply-and-demand system as a fuzzy linear system, we can capture this imprecision and obtain fuzzy equilibrium solutions. The use of fuzzy numbers in this model provides a more realistic and flexible representation of market behavior, particularly when classical crisp models fail to capture the fundamental imprecision in economic data.

Earlier studies, such as [13], attempted to solve such systems using the QR decomposition method, but this approach fails in cases when the system becomes singular. We can see that for system (4), the matrix S, mentioned in [13] is given by

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

which is singular. Hence,the solution can not be obtained by QR Decomposition method. [14] addressed this limitations by proposing a more general solution technique that can handle systems with negative fuzzy parameters and successfully demonstrated its applicability to supply-demand models. In this study, we solve the model using fuzzy Cramer's rule, based on our newly proposed arithmetic operations.

$$\begin{aligned} (0.5, 1, 1.7) \otimes \tilde{x}_1 \oplus (-3, -1, -0.8) \otimes \tilde{x}_2 &= (3, 5, 7) \\ (0.9, 1, 3) \otimes \tilde{x}_1 \oplus (0.2, 1, 2) \otimes \tilde{x}_2 &= (9, 13, 17). \end{aligned} \tag{4}$$

In matrix form, this system can be written as:

$$\tilde{A} \otimes \tilde{X} = \tilde{B},$$

where

$$\tilde{A} = \begin{bmatrix} (0.5, 1, 1.7) & (-3, -1, -0.8) \\ (0.9, 1, 3) & (0.2, 1, 2) \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} (3, 5, 7) \\ (9, 13, 17) \end{bmatrix}$$

The determinant of the fuzzy coefficient matrix is given by,

$$D = \det(\tilde{A}) = [1.18\alpha + 0.82, 12.4 - 10.4\alpha].$$

The modified matrices are obtained by replacing the corresponding columns of $\tilde{A}$ with $\tilde{B}$,

$$\tilde{A}_1 = \begin{bmatrix} \tilde{b}_1 & \tilde{a}_{12} \\ \tilde{b}_2 & \tilde{a}_{22} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} \tilde{a}_{11} & \tilde{b}_1 \\ \tilde{a}_{21} & \tilde{b}_2 \end{bmatrix}.$$

Compute D_1 and D_2 as,

$$\begin{aligned} D_1 &= \det(\tilde{A}_1) = [10.2\alpha + 7.8, 65 - 47\alpha] \\ D_2 &= \det(\tilde{A}_2) = [24.5\alpha - 16.5, 26.2 - 18.2\alpha]. \end{aligned}$$

Finally, the fuzzy solutions are obtained as,

$$\tilde{x}_1 = \frac{D_1}{D} = \left[\frac{10.2\alpha + 7.8}{12.4 - 10.4\alpha}, \frac{65 - 47\alpha}{1.18\alpha + 0.82}\right]$$

$$\tilde{x}_2 = \frac{D_2}{D} = \left[\frac{24.5\alpha - 16.5}{1.18\alpha + 0.82}, \frac{26.2 - 18.2\alpha}{1.18\alpha + 0.82}\right].$$

Since the conditions for the existence of the fuzzy solution are satisfied, we can construct $\tilde{x}_1$ and $\tilde{x}_2$ using the theorem (2.8),

$$\tilde{x}_1 = (0.629, 9, 79.2683)$$
$$\tilde{x}_2 = (-20.122, 4, 31.9512).$$

The fuzzy solution $\tilde{x}_1$ and $\tilde{x}_2$ are shown in Figure (6).

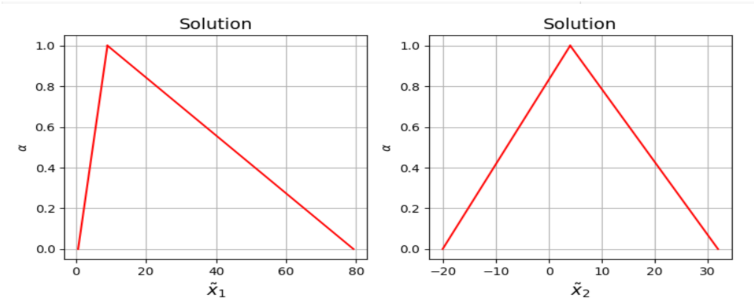

Figure 6: Solutions of System

8 Conclusion

In this paper, we have introduced novel definitions for the four basic fuzzy arithmetic operations on fuzzy numbers. Unlike existing approaches, the proposed method is advantageous that it has the ability to recover the original fuzzy numbers from the result of an operation, thereby ensuring consistency and reversibility. Through illustrative examples involving triangular fuzzy numbers of different types and lengths, the method is shown to be computationally straightforward, and applicable to positive and negative fuzzy numbers, as well as those around zero. This property enhances the reliability of fuzzy arithmetic in practical applications and provides a stronger mathematical foundation for solving fuzzy linear equations of the form $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \otimes \tilde{X} = \tilde{B}$, as shown. The method was also applied to fuzzy system of linear equations, with applications to supply–demand and fuzzy systems demonstrating its practical usefulness. The approach can be extended to further research in fuzzy mathematics and its practical applications. Our study revisits the fuzzy supply and demand model with a fresh perspective. We re-examine the same class of problems using the proposed method. The objective is not only to compare our results with those obtained by [14] but also to investigate whether the method ensures the existence of solutions.

While the focus here has been on triangular fuzzy numbers, which is easily applicable to trapezoidal fuzzy number, future work will aim to extend the proposed method to Gaussian fuzzy numbers. This will allow us to test the generality and robustness of the approach across broader classes of fuzzy sets. Furthermore, the method will be applied to additional real-world problems in economics, engineering, and biology under uncertainty, thereby highlighting its practical relevance and supporting the development of more advanced fuzzy modeling techniques.

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