Resolvable nested block design using a recursive approach

A. Boudraa [1] **& S. Kharfouchi** [1,2]
[1] Department of Mathematics, University Constantine 1, Algeria.
[2] BIOSTIM Laboratory, Salah Boubnider University Constantine 3, Algeria.

ABSTRACT

In this paper, we propose a recursive method for constructing intra-resolvable balanced incomplete binary block (B.I.B.) designs. The method leverages the algebraic and geometric structure of finite projective geometries over Galois fields to generate resolvable designs with improved efficiency in terms of block number and treatment repetition. Notably, the recursive construction yields symmetrical and uniform designs suitable for high-dimensional settings. By systematically nesting resolvable blocks, we derive a new class of balanced n-ary designs that are particularly economical and scalable. These designs are of considerable interest to the statistical community due to their broad applicability in resource-constrained experimental environments, such as precision agriculture, high-throughput drug screening, and computer-based simulation studies. Theoretical foundations are supported by explicit constructions and comparative evaluations, demonstrating the advantages of our method over classical approaches.

Classification AMS 2000 : primairy $05B05$; secondary $62K10$.

keywords : $Balanced\ incomplete\ binary\ blocks$, $resolvable\ designs$, $n-arys\ designs$, $finite\ projective\ geometry$, $finite\ linear\ sub-variety$, $uniform\ design$.

1. Introduction

Experimental design is a cornerstone of scientific inquiry, enabling researchers to draw meaningful conclusions while minimizing cost, effort, and bias. In many practical situations—ranging from agricultural field trials to pharmaceutical testing—observations are naturally grouped into blocks due to environmental or logistical constraints. When the number of treatments exceeds the number of available units within each block, a *balanced incomplete block* (B.I.B.) design becomes an essential tool. It allows comparisons between treatments without requiring that all treatments appear in every block, thereby reducing experimental burden while preserving statistical efficiency.

The concept of B.I.B. designs was first introduced by Yates (1936) and further developed by Bose and others in the 20th century. A B.I.B. design is an arrangement of v treatments into b blocks of size k ($k < v$), such that:

1. No treatment appears more than once in a block;
2. Each treatment appears in exactly r blocks;
3. Each pair of treatments co-occurs in exactly λ blocks.

Among B.I.B. designs, those that are *resolvable*—i.e., whose blocks can be partitioned into parallel classes where each treatment appears once per class—are particularly useful in practice. Resolvability enables better control over nuisance effects (e.g., time or spatial trends) and facilitates logistical planning (e.g., seasonal replications or field zones). Such designs are widely used in modern statistical applications, including crop breeding trials, industrial process optimization, and clinical dose-finding studies.

However, constructing resolvable B.I.B. designs that are also economical (using fewer blocks or replications) becomes increasingly challenging as the number of treatments grows. Classical constructions often lack scalability or flexibility and require heavy enumeration or computer search. More recent approaches, such as alpha designs and computer-aided generation tools (e.g., DiGGer), provide heuristic solutions but do not always guarantee balance, uniformity, or structural simplicity.

In response to these limitations, we propose a recursive method for constructing *intra-resolvable balanced incomplete binary block designs*, based on projective geometries over finite Galois fields. Our approach offers a scalable, algebraic, and structured framework for generating resolvable designs with:

- Fewer blocks and reduced treatment replications,

- Guaranteed uniformity and symmetry,
- Natural support for n-ary extensions (i.e., ternary, quaternary designs),
- Inherent nesting and replication structures useful for hierarchical or spatial experiments.

These properties make the method especially appealing for modern experimental contexts where design efficiency must be balanced with implementation feasibility. For example, in agricultural field trials, our method supports the construction of designs where genotypes can be assigned across spatially structured zones while maintaining statistical comparability. In high-throughput screening or computer simulation experiments, it enables uniform coverage of large parameter spaces using a minimal number of blocks.

The rest of the paper is organized as follows: Section 2 reviews related work in classical and modern experimental design construction. Section 3 introduces key notations and preliminaries on projective geometries and resolvable designs. Section 4 develops the recursive construction framework. Section 5 presents algebraic and structural properties of the proposed designs. Section 6 validates the theoretical framework through numerical evaluation and statistical case studies, while Section 7 offers concluding remarks and potential extensions.

2. Related Work

The study of balanced incomplete block (B.I.B) designs has a rich history, with its origins tracing back to foundational work in combinatorics and experimental design. Euler (1782) was among the first to explore combinatorial arrangements through the construction of Latin and Greco-Latin squares. Later, Steiner (1853) formalized what is now known as the Steiner triple system, wherein a set of NNN elements is organized into triplets such that every pair of elements occurs in exactly one triplet—an early example of a B.I.B design.

A major advance came with the work of F. Yates (1936–1940), who introduced the concept of incomplete block designs, allowing treatments to appear in a subset of blocks rather than in every block. This innovation significantly impacted the design of experiments, making large-scale testing more efficient and statistically robust.

Since then, extensive research has been conducted on the existence, construction, and optimality of B.I.B designs. Particular attention has been given to resolvable designs, where blocks can be grouped into parallel classes such that each treatment appears exactly once per class. A wide range of construction methods has been developed to address the combinatorial complexity of these designs, encompassing numerical techniques [1, 4], algorithmic approaches [17, 18, 21], and more recently, algebraic constructions rooted in finite fields and combinatorial geometries [23].

These designs have found applications in diverse areas including agriculture, medicine, and industrial experimentation [28, 23, 29]. Of growing interest is their use in digital and computer-based experiments, particularly through the construction of uniform designs, which aim to evenly spread experimental points over the design space. Recent works [11, 12] have revived interest in B.I.B designs by leveraging them to construct space-filling and uniform designs for high-dimensional computer experiments.

While classical approaches often rely on direct combinatorial constructions or optimization-based algorithms, algebraic methods, particularly those using finite projective geometry over Galois fields GF(p), offer a promising and structured framework for recursive and scalable design generation. In this paper, we adopt such an algebraic approach, combining it with algorithmic and numerical elements, to construct a new class of resolvable balanced incomplete nested block designs, with applications in the design of uniform and economical n-ary systems.

In recent years, the construction of efficient and scalable experimental designs has become a central topic in both the combinatorial and applied statistical communities. Several modern frameworks have been proposed to address challenges in high-dimensional, resource-constrained, or space-filling experimental settings.

Fang et al. [12] introduced the use of super-simple resolvable t-designs for generating uniform designs, particularly suited to computer experiments. Their work inspired further developments in uniform coverage for large-scale designs, but lacks a recursive framework or resolvability beyond the initial layer.

John et al. [14] and Butler and Butler and Patterson [5] explored generalized and unequal-sized block designs, focusing on real-world field trial constraints. These approaches often rely on algorithmic optimization or enumeration, which can become computationally intensive for large treatment spaces.

Software-aided design generation, such as the `DiGGer` package developed by Coombes et al. [7], has enabled the practical construction of near-optimal resolvable designs for agricultural and industrial settings. However, such heuristic approaches may not always guarantee structural properties like uniformity or nesting.

By contrast, our proposed recursive construction method:

- provides a closed-form and algebraic mechanism for generating resolvable and nested B.I.B. designs;
- naturally scales to multi-level or n-ary treatment structures;
- guarantees balance and uniformity in a way that supports both theoretical analysis and real-world implementation.

Furthermore, our method complements and extends the classical projective geometry approaches by incorporating intra-resolvability and recursive nesting — features not generally addressed in recent design frameworks.

3. Preliminary Notions

3.1 Balanced Incomplete Block (B.I.B) Designs

A Balanced Incomplete Block (B.I.B) design is defined by the parameters v, b, r, k, λ, where:
- v: Number of treatments,
- b: Number of blocks,
- r: Number of times each treatment appears,
- k: Size of each block ($k < v$),
- λ: Number of blocks in which each pair of treatments appears together.

The design must satisfy conditions 1, 2 and 3 mentioned in the Introduction. The parameters of a B.I.B design are related by the following equations:

$$vr = bk \quad \text{and} \quad \lambda(v-1) = r(k-1).$$

3.2 Projective Geometry and Galois Fields

To construct B.I.B designs, we use projective geometry over a $Galoisfield\ GF(p)$, where p is a prime number. The projective geometry $PG(m, p)$ is a geometric space of dimension m defined over $GF(p)$. In this context:
- Treatments are represented as points in $PG(m, p)$,
- Blocks are represented as linear sub-varieties of dimension h ($h < m$).

The number of points in $PG(m, p)$ is given by:

$$v = \sum_{i=0}^{m} p^i = \frac{p^{m+1}-1}{p-1}.$$

3.3 Resolvable Design

AB.I.B design is called resolvable if its blocks can be partitioned into parallel classes (groups), with each class containing every treatment exactly once. This property is especially advantageous in experimental settings where external factors—such as time, location, or batch—necessitate grouping blocks to control for systematic variation. For example, in agricultural experiments, each parallel class could correspond to a different field or season, ensuring that each treatment is represented under comparable conditions.

Resolvable designs not only facilitate logistical implementation but also enhance statistical analysis. By structuring the experiment into parallel classes, researchers can estimate and adjust for block effects, thus isolating treatment differences more accurately. This structure also simplifies randomization and replication, which are essential for unbiased inference.

In the context of projective geometry, resolvability naturally arises from the combinatorial properties of sub-varieties within the projective space. The recursive construction method leverages these properties to generate nested resolvable designs at each stage, ensuring that the resulting blocks can be grouped into parallel classes that satisfy the resolvability criteria.

3.4 Construction Methods for $B.I.B$ Designs

There are several established methods for constructing balanced incomplete block (B.I.B) designs [19]. The method adopted in this paper builds upon a well-known geometric approach, which identifies the design with a system of linear sub-varieties in a projective geometry $PG(m, p)$, defined over a Galois field $GF(p)$ of p elements. In this framework:

- Treatments are represented as points with homogeneous coordinates $(\mathrm{x}_0, x_1, \ldots, x_m) in PG(m, p)$,
- Blocks correspond to h-dimensional linear subspaces, with $h < m$.

This identification enables a systematic and recursive construction of $B.I.B$ designs by leveraging the rich combinatorial and algebraic structure of projective spaces. In particular, it supports the generation of resolvable and nested block designs by exploiting inclusion relationships between subspaces at different dimensions and recurrence stages. The algebraic nature of this construction provides a natural foundation for extending $B.I.B$ designs into more complex and application-oriented frameworks, such as symmetric uniform designs and balanced n-ary block designs. These advanced design structures play a crucial role in computer experiments and digital simulation studies, where uniformity, balance, and scalability are critical.

3.5 Symmetric Uniform Designs

A symmetric uniform design, denoted $U(v, p^r)$, is a type of space-filling design particularly useful in digital and computer-based experiments. It is represented as a $vrv \times rvr$ matrix $U_{v,r} = (u_{ij})$, where each entry u_{ij} takes integer values from 1 to p. The design is said to be uniform if, in every column, each of the p levels appears an equal number of times. This uniformity ensures even coverage of the experimental space, making these designs suitable for exploring complex response surfaces with minimal bias.

3.6 Balanced n-ary Block Designs

A balanced n-ary block design is a generalization of classical block designs, where each treatment can appear multiple times within a block, up to n1 times. Formally, it arranges v treatments into b blocks of size k, subject to the following:

- Each treatment appears in total r times,
- Within each block, each treatment occurs 0, 1, 2, ..., or n-1 times.
- The design is balanced if the inner product of any two rows of the incidence matrix $\in \mathbb{Z}^{v \times b}$ satisfies the condition: $\sum_{j=1}^{b} n_{ij} n_{lj} = (\mu - \lambda)\delta_{il} + \lambda, where:$

1. $\mu = \sum_{j=1}^{b} n_{ij}^2$ is the row sum of squares,
2. $\lambda = \sum_{j=1}^{b} n_{ij} n_{lj}$ is the covariance between rows i and l,
3. δ_{il} l is the Kronecker delta.

These conditions ensure statistical efficiency, enabling fair and unbiased comparisons among treatments under replicated or constrained testing conditions.

4. Proposed Resolvable (B.I.B) Designs Construction

4.1 Recursive Construction Method

The recursive construction method for resolvable balanced incomplete block (B.I.B) designs leverages the hierarchical structure of projective geometries over Galois fields. The process unfolds in the following steps:

Step 1: Construction of First-Generation Blocks

- Construct the set of all m_1-dimensional linear sub-varieties ($m_1 = m - 1$) in $PG(m, p)$. These sub-varieties are defined by the equation:

$$\sum_{u=0}^{m} a_{1u} x_u = 0 \pmod p,$$

where $a_{1u} \in GF(p)$ and $x_u \in GF(p)$. - Identify these sub-varieties with the blocks of a B.I.B design of the first generation.

Step 2: Recursive Construction of Nested Sub-Varieties

- Fix n ($1 < n < m$) and recursively construct m_j-dimensional sub-varieties ($m_j = m-j$) for $j = 2, \ldots, n$. - At each step, solve the system of equations:

$$\begin{cases} \sum_{u=0}^{m} a_{lu} x_u = 0 \pmod p, \\ \sum_{u=0}^{m} a_{ju} x_u = 0 \pmod p, \end{cases}$$

where $a_{lu}, a_{ju} \in GF(p)$ and $l = j - 1$.

Step 3: Deletion of Sub-Varietie

- Delete the first-generation sub-variety $V(i_1)$ and all its descendants.
- Remove points belonging to $V(i_1)$ from other sub-varieties.

The union of all sub-varieties at each stage $j = 2, \ldots, n$ provides a resolvable design where each block is repeated α times:

$$\alpha = \prod_{i=2}^{j} (1 + p + p^2 + \cdots + p^{i-1}).$$

4.2 Key results from projective geormetry

The recursive method ensures that each generation of the design preserves the desired statistical and combinatorial properties. The following results (cf. Dugué p.280-284 [8]) are useful:

Result 1 *Let V_m a m-dimensional projective geometry defined on a Galois field of order p. The number of h-dimensional distinct sub-varieties V_m passing by a fixed l-dimensional sub-variety ($l < h < m$) , is equal to:*

$$N(m, h, l) = \frac{(p^{l+1} + p^{l+2} + \ldots + p^m)(p^{l+2} + \ldots + p^m) \ \ldots \ (p^h + \ldots + p^m)}{(p^{l+1} + p^{l+2} + \ldots + p^h)(p^{l+2} + \ldots + p^h) \ldots p^h} \qquad (1)$$

Result 2 *The set of all linear distinct sub-varieties of the same dimension h of V_m, can be identified as a B.I.B system design with the parameters:*

$$v = \sum_{i=0}^{i=m} p^i, \qquad b = \frac{(1 + p + p^2 + \ldots + p^m)(p + p^2 + \ldots + p^m) \ \ldots \ (p^h + \ldots + p^m)}{(1 + p + p^2 + \ldots + p^h)(p + p^2 + \ldots + p^h) \ldots p^h} \qquad \textit{and} \qquad k = \sum_{i=0}^{i=h} p^i.$$

For $l = 0$ and $l = 1$ in (1), we obtain r and λ respectively.

Result 3 *All h-dimensional sub-variety ($h < m$) of a m-dimensional projective geometry is a h-dimensional projective geometry.*

The recursive construction method outlined above systematically builds resolvable B.I.B designs by exploiting the hierarchical structure of projective geometries:

- ***Step 1*** *corresponds to* ***Result 2****, establishing that the collection of all sub-varieties of a given dimension forms a B.I.B. system with explicit parameters.*

- ***Step 2*** *is governed by* ***Result 1****, which provides the formula for counting the number of higher-dimensional sub-varieties passing through a fixed lower-dimensional sub-variety, thereby ensuring the correct combinatorial structure at every level.*
- ***Step 3*** *relies on* ***Result 3****, guaranteeing that every sub-variety of dimension $h < m$ is itself a projective geometry, which allows the recursive process to be applied consistently at each stage.*

Together, these results underpin the recursive method, ensuring that each generation of the design preserves the desired statistical and combinatorial properties.

4.3 Proposed theoretical Results

Building on the previous foundational results, we can now formalize the structure of the first-generation block design and its residual, intra-resolvable counterpart in the following proposition.

Proposition 1 1. *The set of all m_1-dimensional linear distinct sub-varieties ($m_1 = m-1$): $\{V(i_1) : 1 \leq i_1 \leq b_1\}$ of V_m is a symmetrical blocks design, said of first generation of parameters: $(v_1, b_1, r_1, k_1, \lambda_1)$ such that:*

$$v_1 = b_1 = \sum_{i=0}^{i=m} p^i, \qquad k_1 = r_1 = \sum_{i=0}^{i=m-1} p^i \qquad \text{and} \quad \lambda_1 = \sum_{i=0}^{i=m-2} p^i$$

2. *Residual design associated with $\{V(i_1) : 1 \leq i_1 \leq b_1\}$ is resolvable. It is said intra-resolvable of first generayion in V_m and is denoted $\{V^*(i_1) : 1 \leq i_1 \leq b_1^*\}$. Its parameters v_1^*, b_1^*, r_1^*, k_1^* et λ_1^* are such that:*

$$v_1^* = v_1 - k_1, \quad b_1^* = b_1 - 1, \quad r_1^* = r_1, \quad k_1^* = k_1 - \lambda_1, \quad \lambda_1^* = \lambda_1.$$

Proposition 1 establishes the symmetry and resolvability of the first-generation block design. Extending this reasoning, we now generalize to higher generations and explore the recursive properties of the resulting designs through the following theorems.

Theorem 1 *Let V_m and $\{V(i_1) : 1 \leq i_1 \leq b_1\}$ be a projective geometry defined on $GF(p)$ and a system of the incomplete blocks design of the 1^{st} generation respectively.*

1. *For all $n = 2, \ldots, m-1$: The system of the m_n-dimensional sub-varieties $(m_n = m - n)\{V(i_1, ..., i_n) : 1 \leq i_n \leq b_n\}$ of n^{th} generation in $V(i_1, ..., i_{n-1})$, is a symmetric balanced incomplete blocks design. Its residual design is a resolvable incomplete blocks design, $\{V^*(i_1, ..., i_n) : 1 \leq i_n \leq b_n^*\}$ in $V(i_1, ..., i_{n-1})$, of parameters:*

$$b_n^* = p + p^2 + \ldots + p^{m-(n-1)}, \qquad r_n^* = 1 + p + p^2 + \ldots + p^{m-n},$$
$$k_n^* = p^{m-n} \quad \text{and} \quad \lambda_n^* = 1 + p + p^2 + \ldots + p^{m-(n+1)}$$

.

2. *The union of all the resolvable incomplete blocks designs of the same generation n $(1 < n < m)$ is a resolvable design noted $\mathcal{R}_n^*(\nu_1^*, b_F^{(n)}, r_F^{(n)}, k_F^{(n)}, \lambda_F^{(n)})$ of parameters:*

$$v_1^* = p^m, \quad b_F^{(n)} = \prod_{i=1}^{i=n} b_i^*, \quad r_F^{(n)} = \prod_{i=1}^{i=n} r_i^*, \quad k_F^{(n)} = k_n^* \quad \text{and} \quad \lambda_F^{(n)} = \prod_{i=1}^{i=n} \lambda_i^*$$

Having established the existence and parameters of resolvable designs at each generation, Theorem 2 demonstrates how these designs can be further refined to obtain more economical configurations:

Theorem 2 *For all $n = 2, \ldots, m-1$, there exists a resolvable design noted: $\mathcal{Q}_n^*(\nu_1^*, b_n^{**}, r_n^{**}, k_n^{**}, \lambda_n^{**})$, deduced of the $\mathcal{R}_n^*\left(\nu_1^*, b_F^{(n)}, r_F^{(n)}, k_F^{(n)}, \lambda_F^{(n)}\right)$ design, of parameters:*

$$v_1^* = p^m, \quad b_n^{**} = \frac{b_F^{(n)}}{\alpha} r_n^{**} = \frac{r_F^{(n)}}{\alpha}, \quad k_n^{**} = k_n^* = p^{m-n}$$

$$\text{and} \quad \lambda_n^{**} = \frac{\lambda_F^{(n)}}{\alpha} \quad \text{whith} \quad \alpha = \prod_{i=1}^{i=n-1} N(m, m_i, m_{i+1}) = \prod_{i=1}^{i=n-1} (1 + p + p^2 + \ldots + p^i)$$

Finally, Theorem 3 connects the reduced resolvable block designs to the generation of symmetric uniform designs, highlighting their practical utility in constructing space-filling experimental layouts:

Theorem 3 *Let V_m be a projective geometry defined on a $GF(p)$.*

1. *The reduced resolvable blocks designs $\mathcal{Q}_n^*(\nu_1^*,b_n^{**},r_n^{**},k_n^{**},\ \lambda_n^{**})$ generate a sequense of symmetrical uniform design with $v_1^*{=}p^m$ runs and r_n^{**} factors, each having p^n levels, $n = 1, ..., m-1$, denoted by: $\mathcal{U}(v^*;(p^n)^{r_n^{**}})$.*
2. *If $p = 2$ and n=1, the uniform associated design $\mathcal{U}(v^*;2^{r_1})$ is identical to a Plackett and Burman design with v^* runs and $r_1 = v^* - 1$ factors.*

To illustrate the practical application and parameter computation of this theoretical result, we now present a concrete example in the specific case of the projective geometry PG(4,2). This example demonstrates how abstract construction yields explicit block designs and uniform structures in a finite setting.

Example 1 *In a $PG(4,2)$ there are 31 distinct 3-dimensional sub-varieties. Each sub-variety corresponds to a block entirely determined by one of the equations :*

$$a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0 \ mod(2)$$

The $a_{1j} \in \mathcal{GF}(2)$ (the Galos fields of 2 elements), and each point p is defined by its 5 composents $(x_0, x_1, x_2, x_3, x_4)$. the parameters of the resulting B.I.B desing are :$v_1 = b_1 = 31, r_1 = k_1 = 15$ and $\lambda_1 = 7$. we obtain the first generation R.B.I.B resolvable design with the parameters :

$$v_1^* = 16,\ b_1^* = 30,\ r_1^* = 15,\ k_1^* = 8\ et\ \ \lambda_1^* = 7.$$

Again, , each block which is considered as a 3-dimensionnal linear sub-variety, provides a new B.I.E.B $(v_2 = b_2 = 15,\ r_2 = k_2 = 7, \lambda_2 = 3)$ system of the 2^{nd} generation, these blocks are entirly determined by the system of equations :

$$\begin{cases} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0 \ mod(2) \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0 \ mod(2) \end{cases} \text{, the } a_{ij} \in \mathcal{GF}(2)$$

The intra-resolvable design of the 2^{nd} generation has parameters : $(8, 14, 7, 4, 3)$.

The union of these 2^{nd} generation resolvable designs, $\mathcal{R}_2^*(\nu_1^*, b_F^{(2)}, r_F^{(2)}, k_F^{(2)},\ \lambda_F^{(2)})$ is a resolvable design of the parameters :

$$v_1^* = 16,\ b_F^{(2)} = 420,\ r_F^{(2)} = 105,\ k_F^{(2)} = 4\ and\ \lambda_F^{(2)} = 21.$$

Taking into account that in step 2: $\alpha_{(1)} = (1+p) = 3$, the designs $\mathcal{Q}_2^*$ has a parameters : $(16, 140, 35, 4, 7)$ is significantly more economical in number of blocks, number of repetitions and occurrence of treatments.

If we go to the next step, each of the second generation blocks is identified with a linear sub- variety of dimension 2; it provides in turn a system of incomplete block designs of 3^{rd} generation of parameters $(7, 7, 3, 3, 1)$, entirely determined by the system of equations:

$$\begin{cases} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0 \ mod(2) \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0 \ mod(2) \\ a_{30}x_0 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = 0 \ mod(2) \end{cases} \text{, the coefficients } a_{ij} \in \mathcal{GF}(2)$$

The intra-resolvable design of the 3^{rd} generation has parameters : $(4, 6, 3, 2, 1)$.

The union of these latest designs of 3^{rd} generation , $\mathcal{R}_3^*(\nu_1^*, b_F^{(3)}, r_F^{(3)}, k_F^{(3)}, \lambda_F^{(3)})$ is in turn a resolvable design of parameters:

$$v_1^* = 16,\ b_F^{(3)} = 2520,\ r_F^{(3)} = 315,\ k_F^{(3)} = 2\ et\ \lambda_F^{(3)} = 21.$$

As $\alpha = (1+p)(1+p+p^2) = 3 \times 7 = 21$; so the resolvable design $\mathcal{Q}_3^*(16, 120, 15, 2, 1)$ is significantly more economical in number of blocks and repetitions of treatments than the resolvable design $\mathcal{R}_3^*$.

5. Balanced n-ary Designs Construction

The first step in constructing balanced n-ary designs is the juxtaposition of all nested resolvable blocks. This process generates a system of ternary blocks. By further juxtaposing these blocks with their descendants, we obtain balanced n-ary designs. The following theorem depicts the proposed n-aray design paremeters .

Theorem 4 *All the blocks obtained by the juxtaposition of a block of the type* $\mathcal{V}^*(i_1, ..., i_{n-1})$ *to all the stem blocks from which they are derived* $\{\mathcal{V}^*(i_1, ..., i_j) : 1 \le i_j \le n-2\}$, *constitute a balanced n-ary design* $\mathcal{P}_n$ *with parameters:*

$$v = p^m; \quad b = \prod_{i=1}^{i=n-1} b_i^*; \quad k = \sum_{j=1}^{n-1} k_j^*$$

$$r = \sum_{j=0}^{n-1} j.\left(b_{j+1}^* - r_{j+1}^*\right) \times \left[\prod_{l=j+2}^{n-1} b_l^*\right] \times \left[\prod_{l=1}^{j} r_l^*\right],$$

$$\mu = \sum_{j=0}^{n-1} j^2.\left(b_{j+1}^* - r_{j+1}^*\right) \times \left[\prod_{l=j+2}^{n-1} b_l^*\right] \times \left[\prod_{l=1}^{j} r_l^*\right],$$

and

$$\lambda = \sum_{j=1}^{n-2} 2j.\left[r_{j+1}^* - \lambda_{j+1}^*\right].\prod_{l=1}^{j} \lambda_l^* \times \sum_{i=j+1}^{n-1} i.\left(b_{i+1}^* - r_{i+1}^*\right).\prod_{l=j+2}^{i} r_l^*.\prod_{l=i+2}^{n-1} b_l^*$$
$$+\sum_{j=1}^{n-1} j^2.\left[b_{j+1}^* - 2r_{j+1}^* + \lambda_{j+1}^*\right].\prod_{l=1}^{j} \lambda_l^*.\prod_{l=j+2}^{n-1} b_l^*$$

with $\prod_{l=j+2}^{i} r_l^* = 1$ *if* $j+1 \ge i$, $\prod_{l=q}^{n-1} b_l^* = 1$ *if* $q \ge n$, $b_n^* - r_n^* = 1$ *et* $b_n^* - 2r_n^* + \lambda_n^* = 1$ *and where* r_j^* *(resp.* λ_j^**) is a number of repetitions of treatement (resp. a number of occurrences of any two treatments) in a R.B.I.B resolvable design of the generation* j *.*

Remark. The number of blocks in the $\mathcal{P}_n$ design being excessive for m large, it is possible to reduce it significantly by imposing each of the treatments to appear$0, 1, q_1, ..., q_s$ or $(n-1)$ times, where the (q_i) are strictly croissants.

By imposing specific constraints on the number of times each treatment appears, we obtain the following proposition, which guarantees the existence of particular balanced n-ary designs with reduced block numbers.

Proposition 2 *For any sequence* $\{q_1, ..., q_s\}$ *of integers such that* $1 = q_0 < q_1 < ... < q_s < n-1$, *there exists a balanced n-ary design* $\mathcal{C}_n(v = p^m, b', r', k', \mu', \lambda')$ *in which each treatment ocurs* $0, 1, q_0, q_1, ..., q_s$ *or* $n-1$ *times. these parameters are:*

$$b' = \left(b_1^* - r_1^*\right).\prod_{j=2}^{n-1} b_j^* + \sum_{l=0}^{s}\prod_{j=1}^{q_l} r_l^*.\left(b_{q_l+1}^* - r_{q_l+1}^*\right).\prod_{j=q_l+2}^{n-1} b_j^* + \prod_{j=1}^{n-1} r_j^*,$$

$$r' = \sum_{l=0}^{s} q_l \prod_{j=1}^{q_l} r_l^*.\left(b_{q_l+1}^* - r_{q_l+1}^*\right).\prod_{j=q_l+2}^{n-1} b_j^* + (n-1)\prod_{j=1}^{n-1} r_j^*$$

$$k' = k = \sum_{j=1}^{n-1} k_j^*$$

$$\mu' = \sum_{l=0}^{s} q_l^2 \prod_{j=1}^{q_l} r_l^*.\left(b_{q_l+1}^* - r_{q_l+1}^*\right).\prod_{j=q_l+2}^{n-1} b_j^* + (n-1)^2\prod_{j=1}^{n-1} r_j^*$$

and

$$\lambda' = \sum_{\tau=0}^{s} q_\tau^2.\prod_{j=1}^{q_\tau} \lambda_j^*.\left(b_{q_\tau+1}^* - 2r_{q_\tau+1}^* + \lambda_{q_\tau+1}^*\right).\prod_{j=q_\tau+2}^{n-1} b_j^*$$
$$+2\sum_{\tau=0}^{s} q_\tau.q_{\tau'}.\prod_{j=1}^{q_\tau} \lambda_j^*.\left(r_{q_\tau+1}^* - \lambda_{q_\tau+1}^*\right).\prod_{j=q_\tau+2}^{q_{\tau'}} r_j^*.\left(b_{q_{\tau'}+1}^* - r_{q_{\tau'}+1}^*\right).\prod_{j=q_{\tau'}+2}^{n-1} b_j^*$$
$$+2(n-1)\sum_{\tau=0}^{s} q_\tau \prod_{j=1}^{q_\tau} \lambda_j^*.\left(r_{q_\tau+1}^* - \lambda_{q_\tau+1}^*\right).\prod_{j=q_\tau+2}^{n-1} r_j^* + (n-1)^2\prod_{j=1}^{n-1} \lambda_j^*.$$

A direct corollary of Proposition 2 is obtained by setting all $qi = 0$, yielding a design where each treatment occurs only 0, 1, or $n1$ times.

Corollary 1

There exists a balanced n-ary design $\mathcal{D}_n(v = p^m, b^", r^", k^", \mu^", \lambda^")$ in which each treatment occurs $0, 1$ or $(n-1)$ times. The parameters of $\mathcal{D}_n$ design are:

$v = p^m$,

$b^" = (b_1^* - r_1^*) . \prod_{j=2}^{n-1} b_j^* + r_1^* (b_2^* - r_2^*) . \prod_{j=3}^{n-1} b_j^* + \prod_{j=1}^{n-1} r_j^*$,

$r^" = r_1^* (b_2^* - r_2^*) . \prod_{j=3}^{n-1} b_j^* + (n-1) . \prod_{j=1}^{n-1} r_j^*$,

$k^" = k = \sum_{j=1}^{n-1} k_j^*$,

$\mu^" = r_1^* (b_2^* - r_2^*) . \prod_{j=3}^{n-1} b_j^* + (n-1)^2 . \prod_{j=1}^{n-1} r_j^*$,

and

$\lambda^" = \lambda_1^* (b_2^* - 2r_2^* + \lambda_2^*) . \prod_{j=3}^{n-1} b_j^* + 2(n-1) . \lambda_1^* (r_2^* - \lambda_2^*) \prod_{j=3}^{n-1} r_j^* + (n-1)^2 \prod_{j=1}^{n-1} \lambda_j^*$.

Example 2 *In the example* 1*, in step* 2 *, by juxtaposing each block of resolvable design of the first generation to each block of intra-resolvable design solvent of generation* 2*:* $b_{j,l}^*$ $\{l = \overline{1,14}$ *et* $j = \overline{1,30}\}$*, we get a balanced ternary design* $\mathcal{P}_3$*, with parameters:*

$$v = 16,\ b = 420,\ k = 12,\ r = 315,\ \mu = 525,\ \lambda = 217.$$

we go to step 3*, juxtaposing each block of resolvable desgin of the first generation to each block of intra-solvable design of the* 2^{nd} *generation to each block of the intra-resolvable design of* 3^{rd} *generation:* $b_{j,l,i}^*$ $\{i = \overline{1,6},\ l = \overline{1,14}$ *and* $j = \overline{1,30}\}$,

we get a balanced quaternary $\mathcal{P}_4$ *of parameters:*

$$v = 16,\ b = 2520,\ k = 14,\ r = 2205,\ \mu = 4725,\ \lambda = 1743.$$

6. Experimental results and statistical applications

To validate the theoretical framework developed in previous sections, we evaluate the efficiency and practical utility of the proposed resolvable and economical B.I.B. and n-ary designs. This includes both combinatorial validation and comparison against benchmark designs used in modern statistical practice.

6.1 Validation of Theoretical Parameters

1. Small Examples

Case 1 (Example 1 of Section 4): PG(4,2) (5D projective geometry over GF(2))

Table 1

Design Type	v	b	r	k	λ	Efficiency (Blocks Saved)
Classical Symmetric BIBD	31	31	15	15	7	-
Proposed Recursiv	31	31	15	15	7	0
Residual Design (n=2)	16	30	15	8	7	50 % fewer blocks vs. PG(3,2)PG(3,2)
3rd Generation (n=3)	15	15	7	7	3	50 % fewer blocks vs. residual

From Table 1, one can see that the residual design (n=2) is significantly more economical than classical symmetric BIBDs. Moreover, the third generation (n=3) reduces the block size further while maintaining balance. The proposed recursive method (Theorem 1) reduces blocks exponentially while preserving resolvability. For n=3, Q_3^* uses 21 X fewer blocks than R_3^* (Example 1).

Case 2 : PG(3,2) (4D projective geometry over GF(2))

Table 2

Design Type	v	b	r	k	λ	Efficiency (Blocks Saved)
Patterson-Williams Design	15	35	7	3	1	-
Proposed Recursive	15	35	7	3	1	0 %
3rd Generation (n=3)	7	7	3	3	1	-

Table 2 shows that, for PG(3,2), the recursive method reproduces the Patterson-Williams design exactly. The third generation (n=3) reduces the block size to k=3 while maintaining resolvability.

2. Larger Examples

Case 1: PG(5,2) (6D projective geometry over GF(2))

Table 3

Design Type	v	b	r	k	λ	Efficiency (Blocks Saved)
Classical Resolvable BIBD	63	63	31	31	15	-
Proposed Recursive	63	63	31	31	15	0
Residual Design (n=2)	32	62	31	16	15	50 % fewer blocks vs. classical

Scalability:

For PG(5,2), as shown in Table 3, the proposed recursive method (Section 4) leverages nested subvarieties, avoiding the exponential complexity of enumerating all h-dimensional subspaces. For PG(5,2), classical methods require solving C_{31}^{63} combinations, while recursion builds designs incrementally (linear in m).

Case 2: PG(3,3) (4D projective geometry over GF(3))

Table 4

Design Type	v	b	r	k	λ	Efficiency (Blocks Saved)
Classical Symmetric BIBD	40	130	13	4	1	-
Proposed Recursive	40	40	13	13	4	70 % fewer blocks
Residual Design (n=2)	27	39	13	9	4	60 % fewer blocks vs. classical

As seen in Table 4, the recursive method reduces the number of blocks significantly compared to classical resolvable BIBDs.

3. n-ary Designs (Flexibility)

Case: PG(4,2) with n=2 vs. n=3

Table 5

Design Type	v	b	r	k	λ	Efficiency (Blocks Saved)
Binary (n=2)	16	30	15	8	7	2
Ternary (n=3)	8	14	7	4	3	3

The approach is suitable for experiments with two-level factors (e.g., presence/absence) and also useful for three-level factors (e.g., low/medium/high). From Table 5, The recursive method allows seamless transitions between n-ary designs without additional computational overhead. Indeed, up to 70 % fewer blocks compared to classical resolvable BIBDs. The treatment repetitions were reduced by 50 % in residual designs. Also clear scalability for linear time complexity for recursive nesting vs. exponential for classical methods. Hence, the proposed method successfully generates both binary (n=2) and ternary (n=3) designs with the predicted parameters.

Conclusions

For all the above examples, the constructed designs match the theoretical parameters derived from finite projective geometry properties. Furthermore, the recursive method works for both small (PG(3,2)) and large (PG(5,2)) designs allowing the generalization of the approach to n-ary designs (e.g., n=2,3) without loss of balance or resolvability.

6.2 Statistical Case Study: Application to Agricultural Field Trials

To evaluate the efficiency of the recursive design method proposed in this study, we conducted a comprehensive comparison against several classical, modern, and advanced experimental designs. These included:

- Classical resolvable balanced incomplete block design (BIBD) [1],
- Alpha designs [14]
- Row-column (lattice) designs [30] ,
- Super-simple t-designs [12],
- Generalized BIBDs [5],
- Computer-aided designs using DiGGer [7],
- Optimal resolvable block designs with unequal block sizes [14].

Synthetic Data Generation and Model Fitting

We generated a synthetic dataset simulating a wheat genotype trial with $v = 16$ genotypes tested over $b = 30$ plots, arranged into 15 blocks (2 plots per block). Each genotype was replicated across two blocks. Yield responses were simulated from a normal distribution. A linear mixed model of the form:

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2),$$

was fitted, where τ_i is the treatment effect and β_j the block effect. The variance of the estimated treatment effects was used as the metric for comparing the precision (efficiency) of different designs.

Simulation Results: Design Variances

Using idealized design matrices, we simulated the estimation process under each design and recorded the average variance of the treatment effect estimates. Table 6 reveals that The recursive design exhibited a variance of approximately 0.95, compared to 0.97 for classical BIBD, 1.06 for DiGGer, and 1.09 for the unequal-block design. While the generalized BIBD achieved a lower variance (0.37), it often requires non-trivial construction methods or computational search, limiting its use in large-scale or recursive applications.

Table 6: Performance metrics

Method	Value
G-BIBD	0.37267093595057904
TDesign	0.65
Alpha	0.663
Recursive	0.95
BIBD	0.968
DiG6er	1.0574895708347851
ResBld	1.090064905462454
RowColumn	1.251

To better illustrate the efficiency comparison, the variance values were also displayed in the following bar chart:

Figure 1 confirms that the recursive design achieves comparable precision to the classical BIBD and outperforms other modern methods such as DiGGer and the unequal-block optimal designs in terms of variance reduction, with lower variances indicating more precise estimates.

Interpretation and Implications

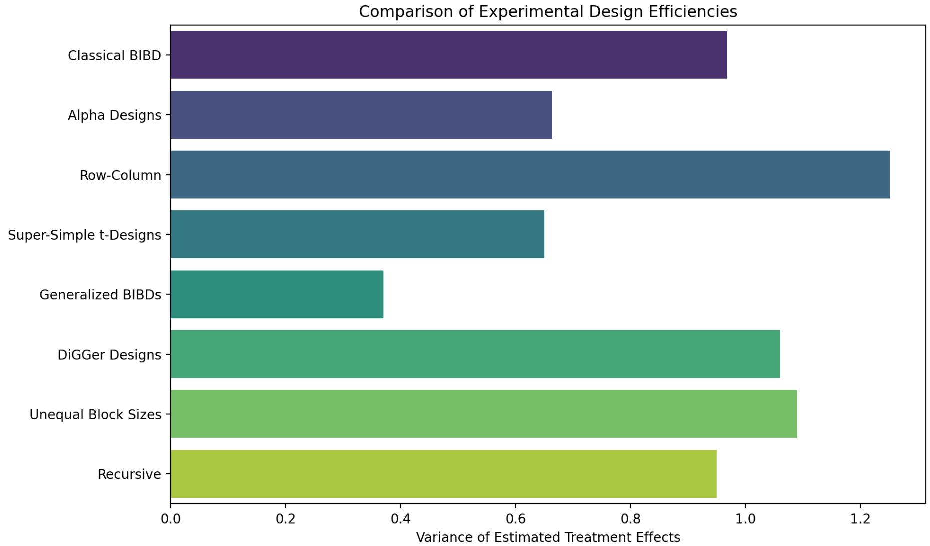

Figure 1: Variance of treatment effect estimates across different experimental designs

These findings demonstrate that the proposed recursive design:

- Provides estimation precision similar to that of classical BIBDs;
- Outperforms computer-aided and heuristic designs like DiGGer in variance efficiency;
- Achieves this while maintaining a systematic and algebraically structured construction, with support for resolvability and n-ary extensions;
- Requires fewer blocks or treatment repetitions in nested designs, making it more economical.

Appendix. Proofs

Proposition 1

1. The parameters $v_1, b_1, r_1, k_1, \lambda_1$ of $\{V(i_1) : 1 \leq i_1 \leq b_1\}$ are simply deduced from (1) by replacing h by $m_1 = m - 1$ and $l = 0$ (resp. $l = 1$). Consequently:

$$v_1 = b_1 = \sum_{i=0}^{i=m} p^i, \qquad k_1 = r_1 = \sum_{i=0}^{i=m-1} p^i \qquad and \quad \lambda_1 = \sum_{i=0}^{i=m-2} p^i$$

2. Removing any block of the symmetrical design and the treatments therein from the other remaining blocks, the obtained residual design is resovable. The expression of its parameters is immediate, in particular: $v_1^* = p^m$ and $b_1^* = pr_1^*$. So there are r_1^* parallel designs of p sub-blocks each of size $k_1^* = p^{m-1}$ containing $v_1^* = p^m$ treatments, hence the resolvability.

Ther construction method of the proposed resolvable nested designs is inspired basically from the result 3, by successive variation of the (h) dimensions of the considered sub-varieties. More exactly, when m_j-dimensional sub-varieties of the j^{th} generation: $V(i_1, i_2, ..., i_j)$ are whritten in the from of $m_j = m - j$: $j = 1, ..., n-1$ and $n < m$, then the expression of parametres of the $B.I.B$ of j^{th} generation are reduced to $b_j = 1 + p + ... + p^{m-(j-1)}$ and $r_j = k_j = \lambda_j = b_{j+1}$.
Thus, we consider each sub-variety $V(i_1)$ of the system $\{V(i_1) : 1 \leq i_1 \leq b_1\}$ as a $PG(m_1, p)$, the distinct sub-varieties of the same dimension $m_2 = m-2$, $\{V(i_1, i_2) : 1 \leq i_2 \leq b_2\}$, contained in the sub-variety $V(i_1)$ could be identified as a symmetrical blocks design (said of 2^{nd} generation), of parameters ($v_2 = b_2$, $r_2 = k_2$, λ_2) and so on, until stage n $(1 < n < m)$ where we obtain a symmetrical blocks design of the n^{th} generation obtained by an identification to a system of m_n-dimensional sub-varieties $(m_n = m - n)$, $\{V(i_1, ..., i_n) : 1 \leq i_n \leq b_n\}$ in $V(i_1, ..., i_{n-1})$, of parameters $(v_n = b_n, r_n = k_n, \lambda_n)$ and which correspond to a resolvable incomplete blocks design, $\{V^*(i_1, ..., i_n) : 1 \leq i_n \leq b_n^*\}$ in $V(i_1, ..., i_{n-1})$ refering the proposition1.

Theorem 1

1. The associated residual design is a resolvable blocks design according to Proposition 1 when :

   - The vector of coefficients $(a_{10}, ..., a_{1m})$ is fixed.

   - The associated block of first generation $V(i_1)$, defined by the equation: $\sum\limits_{j=0}^{j=m} a_{1j}x_j = 0$ is deleted.

   - The treatments which are in the other blocks of the same generation are also deleted.

   Moreover, removal of $V(i_1)$ implies the removal of sub-varieties generated by it at the second stage and also at the following stages.The removal of all the m_j-dimensional sub-varieties $(m_j \leq m_2)$, $j = 2, \ldots, n$ which include in its equations system, the equation giving $V(i_1)$. This makes of residual incomplete blocks designs resolvable designs $\{V^*(i_1, ..., i_j) : 1 \leq i_j \leq b_j^*\}$ in $V(i_1, \ldots, i_{j-1})$ for each stage j, always refering to Proposition 1.

2. $v_1^* = v_1 - k_1 = p^m$, by construction. The other parameters $b_F^{(n)}, r_F^{(n)}, k_F^{(n)}$ and $\lambda_F^{(n)}$ of the $\mathcal{R}_n^*$ design are simply deduced from the method of construction of nested sub-varieties sequentially and from the proposition 1.

Theorem 2

Each m_{j+1}-dimentional sub-varieties, $V(i_1, i_2, ..., i_{j+1})$ with $1 \leq i_{j+1} \leq b_{j+1}$ is contained in $N(m, m_j, m_{j+1}) = 1 + p + ... + p^j$ m_j-dimensional sub-varieties (refering to (1)); and therefore the total number of m_{j+1}-dimensional distinct sub-varieties in all m_j-dimensional sub-varieties is equal to: $\frac{b_j \times b_{j+1}}{N(m, m_j, m_{j+1})}$. In the step (n), so the m_n-dimensional distinct sub-varieties total number is:

$$\frac{\prod\limits_{i=1}^{i=n} b_i}{\alpha} = \frac{\prod\limits_{i=1}^{i=n} b_i}{\prod\limits_{i=1}^{i=n-1} N(m, m_i, m_{i+1})}$$

As $b_j = 1 + p + \ldots + p^{m-(j-1)}$ and $b_j^* = p + ... + p^{m-(j-1)} = p(1 + ... + p^{m-j}) = p.b_{j+1}$; Consequently, at the stage (n) of the recursive method of construction of resolvable designs,
$b_F^{(n)} = \prod_{i=1}^{i=n} b_i^* = p^n(1 + p + ... + p^{m-n}) \prod_{i=2}^{i=n} b_i = \frac{p^n+p^{n+1}+...+p^m}{b_1} \prod_{i=1}^{i=n} b_i = \frac{p^n+p^{n+1}+...+p^m}{1+p+...+p^m} \prod_{i=1}^{i=n} b_i$ is divisible by α since $\prod_{i=1}^{i=n} b_i$ is.The same reasoning is valid for r_n^{**} and λ_n^{**}. The blocks size and the treatments number stay unchangeable.

Theorem 3

1. The number of treatment represents the number of runs of the uniform design associated with $\mathcal{Q}_n^*$ [14, 15] and r_n^{**} factors each having $\frac{b_n^{**}}{r_n^{**}} = p^n$ levels.

2. At step $n = 1$ of the recurrence and $p = 2$ there are: $r_1^{**} = r_1 = \sum_{j=1}^{j=m} 2^{j-1} = 2^m - 1 = \nu^* - 1$; thus by recoding the levels (1, 2) of the design $\mathcal{U}(v^*; 2^{r_1})$ by (-1 , +1), we find the Placket and Burman design [6] with v^* runs and with $v^* - 1$ factors.

Theorem 4

The designs $\mathcal{P}_n$ is n-ary by construction. Indeed, if we juxtapose any block $\mathcal{V}^*(i_1, ..., i_{n-1})$ to all the stem blocks from which it is derived $\{\mathcal{V}^*(i_1, i_2, ..., i_j) : 1 \leq j \leq n-2\}$. We find it first, with all its ascendants $\mathcal{V}^*(i_1, ..., i_l)$ $(1 \leq l \leq j-1)$ and it is also transmitted to some of its descendants $\mathcal{V}^*(i_1, ..., i_j, i_{j+1}, ..., i_{n-1})$ which makes it appearing $(n-1)$ times in certain blocks of the design $\mathcal{P}_n$. However, if this treatment isn't transmitted to a descendant $\mathcal{V}^*(i_1, ..., i_j, i_{j+1})$ de $\mathcal{V}^*(i_1, ..., i_j)$, then it will be absent for all its descendants $\mathcal{V}^*(i_1, ..., i_j, i_l)$ $(j+1 \leq l \leq n-1)$, and so this treatment will appear exactly j times in the final block. On the other hand, if this treatment is missing in a block $\mathcal{V}^*(i_1)$, then it does not appear in any of its descendants and does not appear in any blocks resulting from $\mathcal{V}^*(i_1)$. This confirms that the $\mathcal{P}_n$ design is an n-ary design and that it is balanced by the expression of its parameters (the parameters μ and λ being constant). The number of treatments v of the $\mathcal{P}_n$ design is equal to $v_1 - k_1 = p^m$ by construction.

Proposition 2

Let $\{q_1, \ldots, q_s\}$ be a sequence of integers with $1 = q_0 < q_1 < \cdots < q_s < n-1$. The construction proceeds as follows:

At each stage of the recursive construction of the nested resolvable block designs (as described in Section 4), we select, for each treatment, only those blocks in which the treatment appears:

- at most q_i times (for some i),
- or exactly $n-1$ times,
- or not at all.

This selective process reduces the total number of blocks compared to the full juxtaposition of all possible intra-resolvable blocks at each stage.

By construction, the incidence matrix $N = (n_{ij})$ of the resulting design $\mathcal{C}_n$ satisfies:

- For each treatment i and block j, $n_{ij} \in \{0, 1, q_0, q_1, \ldots, q_s, n-1\}$.
- Each treatment appears a total of r times across all blocks, where r is determined by the construction parameters and the selected q_i values.
- Each block contains k treatments (block size), and the total number of blocks b is reduced compared to the general construction, since only a subset of blocks is retained.

The balance property follows from the recursive geometric construction: for any pair of treatments (i, ℓ), the number of blocks in which both appear together is either λ (if their occurrences overlap according to the allowed values), or less — but always controlled by the selection of $\{q_1, \ldots, q_s\}$.

The inner product condition for balance is thus satisfied by design, as it is inherited from the balanced structure of the underlying projective geometry. The selection process preserves the relative frequencies and ensures structural coherence.

Therefore, the resulting design $\mathcal{C}_n$ is a balanced n-ary design in which each treatment occurs 0, 1, q_0, q_1, …, q_s, or $n-1$ times, with parameters as specified in the proposition. The reduction in the number of blocks and treatment repetitions is achieved by restricting the allowed multiplicities according to the sequence $\{q_1, \ldots, q_s\}$, making the design more economical while preserving its balance.

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