PARAMETRIC CONVERGENCE IMPLIES PROJECTIVE CONVERGENCE IN THE DUAL SPACE OF A FUNCTION SPACE

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ABSTRACT

Enormous work had been done on dual space of a sequence space. Also a massive corpus of work has been done in convergences of different kinds of sequence and sequence spaces. An account of all these can be found out in Cook,1 , chapter 10 P.272-326 . Later on dual space was extended to the case of function spaces. In this concern convergences has also been extended to the case of function spaces by some students of the school of mathematics. The role of dual space of a function space was also observed by some of the researcher of this field in the study of different discipline.

In this paper we have used the notion of dual space of a function space to study the behavior of convergences in different suitably defined function spaces .That is we took the pain to establish some of the results on different kind of convergences for function spaces using the notion of a dual space of a function space. Moreover in course of extending some of the results we tried and got success in observing that if a result is true for a particular function space then it also stands good for another some of the function spaces.

We also observed that the technique of establishing the results for function spaces are quite different from those of the technique of establishing the results to the case of sequence space .As a matter of fact the main reason behind it is the fact that in case of sequence space we had to deal with integers whereas in the case of function or function space, we always play with the continuous variables.

The object of this paper is in fact to establish some of the results to show that parametric convergence implies projective convergence in the dual space of a function space.

KEYWORDS: Linear Space, Sequence Space, Function Space, Dual Function Space, Perfect Function Space, Normal Function Space, Regular Function Space, Convergence Closed Function Space, Parametric Convergent, Parametric Limit, Projective Convergence Projective Limit.

INTRODUCTION

LINEAR SPACE: A structure of linear space on a set 'V' is defind by the two maps:
(a)(x,y)→x+y of V×V into V and is said to be vector addition.(b) **(a,x)→ax** of K×V into V and is said to be scalar multiplication.
The above two maps are assumed to satisfy the following conditions:
(i) x + y = y + x for every x,y in V .**(ii)x + (y + z) = (x + y) + z** for every x,y,z in V .**(iii)**There exits an element 0 in V such that **x + 0 = 0 + x = x** for every x in V.(iv)For every element x in V there exists an element denoted by – x such that **x + (- x) = (-x) + x = 0** for every x in V (v) **a (x + y) = ax + ay** for every a in K and all x,y in V.(vi) **(a + b) x = ax + bx** for every a,b in K and all x in V.(vii) **(ab) x = a (bx)** for every a,b in K and all x in V .(viii) **1x = x** for every x in V. Whenever all the above axioms are satisfied , we say that V is a linear space (or a vector space) over field K.Now if K be the set of all real numbers then V is called a real linear space and similarly if K stands for the set of all complex numbers then V is called a complex linear space . Here every element of V is called a vector and every element of K is called a scalar. The zero vector 0 is unique and called the zero element or the origin in V.

SEQUENCE SPACE :A linear space whose elements are sequences is called a sequence space .Thus a set V of sequences is a sequence space if , it contains the origin and for every x.y in V and for every scalar **α** , x + y and **α**x are in V .
FUNCTION SPACE: A linear space whose elements are functions is called a function space. Thus a set V of functions is a functions space if it contains the origin and for f , g in V and for every scalar α , f + g and **α**f are in V . Here we consider only real functions of real variables.so α is taken to be real scalar so that our purpose is served.

Definitions of some special function spaces are being given below making the use of which some results have been established.
Moreover, the integration has been taken through in Lebesgue sense in the interval $[0 , \infty)$. We denote the set $[0 , \infty)$ by E .
Γ : It denotes the space of all convergent and bounded functions.

L_∞ : It denotes the space of all functions f such that $|\mathbf{f(x)}| < \mathbf{K}$ for almost all $x \geq 0$ where K is constant .

L_1 : It denotes the space of integrable functions , that is L_1 is the space of all functions f such that $\int_E |\mathbf{f(x)}|\,\mathbf{dx} < \infty$

ζ : It denotes the space of all functions continuous and bounded in $[0 , \infty)$. Clearly $\zeta < L_\infty$

ϕ : Let $E = [0. \infty)$. Let E^1 be a subset of E such that $m(E^1)$ is finite . Then the set of all functions f such that f(x) is finite and bounded for almost all x in E1 and is zero in the compliment of E1, is defined to be the space of finite functions and is denoted by **ϕ**.

DUAL SPACE OF A FUNCTION SPACE: α^* of a function space α is the space of all functions off such that $\int_E |\mathbf{f(x)g(x)}|\mathbf{dx} < \infty$ for every function g(x) in α . Also α^* is a function space .Also **$\Gamma^* = L_1$; $L_\infty^* = L_1$; $\zeta^* = L_1$; $L_1^* = L_\infty$** (We refer to Sharan , (1)) .

PERFECT SPACE: A function space α is said to be perfect when $\alpha^{**} = \alpha$. Also L1, L_∞ are perfect. [See Sharan , (1)] .

NORMAL FUNCTION SPACE:A function space α is called normal if for f is in α such that $|g(x)| \leq |f(x)|$ impliesgisinα[SeeSharan(1)].
By earlier work we can see that , every perfect space is normal .Thus it is clear that L_∞ , L_1 are normal function space as these are perfect .[See Sharan , (1)].

REGULAR FUNCTION SPACE: If , with a definition of convergence and limit , every family $f_t(x)$ in α , which has defined limit and also a t – limit , is such that these two limits are equal for almost all $x \geq 0$,then α is said to be regular function space under the defined convergence .

CONVERGENCE CLOSED FUNCTION SPACE: If , with a definition of convergence when parametric limit of every convergent family in a function space α , is itself in α .

PARAMETRIC CONVERGENT (or t – convergent) : Let $f_t(x)$ be a family of functions of x defined for all t in $[0 , \infty)$, where t is a parameter . If to every $\epsilon > 0$, there corresponds a positive number $T(\epsilon)$, idependent of x , such that , for almost all $x \geq 0$,$|\mathbf{f_t(x) - f_{t^1}(x)}| \leq \epsilon$,for all t , $t^1 \geq T(\epsilon)$, then the family $f_t(x)$ is said to be parametric convergent (t – cgt). [See Sharan , (1)] .

PARAMETRIC LIMIT (t – limit) : If , to given any $\epsilon > 0$, there corresponds a number $T(\epsilon)$, independent of x , such that for almost all $x \geq 0$, $|\mathbf{f_t(x) - \psi(x)}| \leq \epsilon$ for all $t \geq T(\epsilon)$, then $\psi(x)$ is called the parametric limit (t–limit) of $f_t(x)$ and we write t–limit of $f_t(x) = \psi(x)$**.** Here we observe that any function equal to $\psi(x)$, for almost all $x \geq 0$, is also a t–limit of $f_t(x)$.Therefore when we say that $\psi(x)$ is the parametric limit (t-limit) of $f_t(x)$, we mean that $\psi(x)$ is a t-limit of $f_t(x)$ and all functions equivalent to $\psi(x)$ in $[0,\infty)$ are t-limits of $f_t(x)$.[A function θ is said to be equivalent to $\psi(x)$ in $[0,\infty)$. when $\theta(x) = \psi(x)$ almost everywhere in $[0,\infty)$.

PROJECTIVE CONVERGENCE (or αβ-convergence or p-convergence) : Let $\alpha^* \supseteq \beta$ and $\mathbf{F_g(t)} = \int_E \mathbf{f_t(x)g(x)dx}$,Where $f_t(x)$ is in α and g(x) is in β then if $F_g(t)$ tends to a definite finite limit as t – tends to ∞ for every g(x) in β then we say that $f_t(x)$ is projective convergent (or p-convergent) relative to β , or $f_t(x)$ is $\alpha\beta$-convergent and $f_t(x)$ is simply called p-convergent in α or α-convergent when $\beta = \alpha^*$.
"A necessary and sufficient condition for αβ- convergence of $f_t(x)$ is that to every g in β and to every $\epsilon > 0$, there corresponds a positive number $T(\epsilon,g)$ such that , for all t , $t^1 \geq T(\epsilon,g)$, $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq \epsilon$."

PROJECTIVE LIMIT [p – limit or αβ - limit] : A function ψ , in α or outside α ,is called a projective limit (p-limit) of $f_t(x)$ in α relative to β and we write $\psi(x) = \alpha\beta$ - limit of $f_t(x)$ when (i) $\int_E |\mathbf{g(x)\psi(x)}|\mathbf{dx} < \infty$ for every g in β , and (ii) $\lim_{t\to\infty} \int_E \mathbf{f_t(x)g(x)\,dx} = \int_E \mathbf{\psi(x)g(x)dx}$ for every g in β .

When $\beta = \alpha^*$, ψ is called a projective limit (p - limit of $f_t(x)$) in α and we write , $\psi(x) = \alpha$ – limit of $f_t(x)$. Different $\alpha\beta$ – limits of $f_t(x)$ can differ only in a set of x of measure zero. Hence when we say that $\psi(x)$ is the $\alpha\beta$ – limits of $f_t(x)$ we mean that $\psi(x)$ is an $\alpha\beta$-limit of $f_t(x)$ and other $\alpha\beta$-limit of $f_t(x)$ are equivalent to $\psi(x)$. It follows from the definitions that every $\alpha\beta$ – limit belong to β^*.

STRONG PROJECTIVE CONVERGENCE (or strong p-convergence): Let $\alpha^* \supseteq \beta$ and $f_t(x)$, t running through $[0,\infty)$, is in α satisfies the condition that to every $\epsilon > 0$ and every p-bounded set U in β corresponds a positive number $T(\epsilon, U)$ such that

$|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq \epsilon$ for every g in U and all t , $t^1 \geq T(\epsilon, U)$ then $f_t(x)$ is said to be strong projective convergent (or strong p-convergent) relative to β or strong $\alpha\beta$- convergent .When $\alpha^* = \beta$ then we say that $f_t(x)$ is strong projective convergent (or strong p-convergent) in α or strong α-convergent . Now letting U consists of only one function, we see from the definition that strong $\alpha\beta$-convergence $\Rightarrow$ $\alpha\beta$-convergence.

STRONG PROJECTIVE LIMIT (or strong p-limit) :A function $\psi(x)$,in or outside α , is called the strong projective limit of $f_t(x)$ in α relative to β and we show this fact by writing , $\psi(x)$ = strong $\alpha\beta$-limit of $f_t(x)$, when $\int_E \mathbf{g(x)\psi(x)dx} < \infty$ for every g in β and To every projective bounded set U in β , and to every $\epsilon > 0$, there corresponds a number $T(\epsilon,U)$ such that for every g in U , $|\int_E \mathbf{g(x)\{f_t(x) - \psi(x)\}dx}| \leq \epsilon$ for every t $\geq T(\epsilon)$ **.** When $\alpha^* = \beta$ then we say that $\psi(x)$ is the strong projective limit of $f_t(x)$ in α and we write $\boldsymbol{\psi(x) = \alpha}$**-limit of** $\mathbf{f_t(x)}$ [cooke, (1) P-305] Different $\alpha\beta$-limits of $f_t(x)$ can differ only in the sets of x of measure zero.Hence when we say that $\psi(x)$ is the strong $\alpha\beta$-limit of $f_t(x)$ then we understand that $\psi(x)$ is an strong $\alpha\beta$-limit of $f_t(x)$ and other strong $\alpha\beta$-limits of $f_t(x)$ are equivalent to $\psi(x)$.

SMOOTHLY PROJECTIVE CONVERGENCE [or p-convergent(S)] : Let $\alpha^* \supseteq \beta$, F(x,t) be in α and g(x,r) be in β . Let $\boldsymbol{\psi_g(t,r)} = \int_E \mathbf{F(x,t)g(x,r)dx}$ .Now if to every $\epsilon > 0$ and to every g(x,r) in β there corresponds a positive number $T(\epsilon,g)$, independent of r ,such that for almost all $r \geq 0$ | $\boldsymbol{\psi_g(t,r)}$ - $\boldsymbol{\psi_g(t^1,r)}$ |= $|\int_E \mathbf{\{F(x,t) - F(x,t^1)\}g(x,r)dx}| \leq \epsilon$ **,** to every t , $t^1 \geq T(\epsilon, g)$, then we say that F(x,t) in α is smoothly projective convergent (or p-convergent (S)) relative to β, or $\alpha\beta$-convergent (S) ; When $\beta = \alpha^*$ then we say that F(x,t) is α-convergent (S) or α-smoothly convergent .
If there be no chance of confusion then we write $\mathbf{F_t(x)}$ instead of $\mathbf{F(x,t)}$ **.** That is $\mathbf{F(x,t)} \equiv \mathbf{F_t(x)}$ .Similarly $\mathbf{g(x,r) \equiv g_r(x)}$ **.**

INTEGRABLE FUNCTION:The function f(x) is said to be integrable (L) or summable on the set E if the integral $\int_E \mathbf{f(x)dx}$ exists and finite .That is $\int_E \mathbf{f(x)dx} < \infty$ .Since , if f is integrable (L) then |f| is integrable(L) [See Natanson ,(1) chapter 5]
Thus clearly, f(x) belongs to L_1. [See Natanson, (1) chapter 5 , Rudin ,(1) chapter 10 , P-243]

CONVERGENT FUNCTION: A function f(x) which is (i) essentially bounded in $[0 , \infty)$, and (ii) tends to a definite finite limit as x tends to ∞ is called a convergent function.

MEASURABLE SET: A bounded set E is said to be measurable if the outer and inner measures are equal .That is when, $\mathbf{m^*E = m_*E}$ (See Natanson ,(1), P-64)

MEASURE OF THE SET E: The common value of these two measures that is the common value of m_*E and m^*E is called the measure of the set E is designated by mE. Hence $\mathbf{mE = m_*E = m^*E}$
Since the above concept of defining measure of the set E is due to Lebesgue , so sometimes we call E " measurable in the Lebesgue sense " or more briefly "measurable (L)" .

If the set E is non – measurable , it is impossible to take about its measure , and the symbol mE is meaningless In particular , we consider all unbounded sets non measurable. [See Natanson ,(1) chapter 3]

PROJECTIVE BOUNDED SET (p-bounded set) If $\alpha^* \supset \beta$ and if $|\int_E \mathbf{f(x)g(x)dx}| \leq \mathbf{K(g)}$ for every f in set X in α and every g in β , where K(g) is a positive constant depending on g , then we say that the set X in α is projective bounded (or p-bounded) relative to β , or $\alpha\beta$-bounded . When $\beta = \alpha^*$, we say that X is projective bounded (p-bounded) in α , or α-bounded . If $\alpha^* \supset \beta$ and we take a set X in α to be the family $f_t(x)$, with t running through $[0,\infty)$, then we say that $f_t(x)$ is $\alpha\beta$-bounded if , $|\int_E \mathbf{f_t(x)g(x)dx}| \leq \mathbf{K(g)}$ for every t in $[0,\infty)$ and every g in β.

In this section we establish some of the results with reference of the notions given in the above section.
Theorem (2.3, I): If a family of functions in a function space Γ^{**} be parametric convergent then it is $\Gamma^{**}\Gamma^*$-convergent.

Proof: Let ft be a family of functions in function space Γ^{**}. Also let f_t be parametric convergent in Γ^{**}.Then to every $\epsilon > 0$,there exists a positive number $T(\epsilon)$, independent of x , such that , for almost all $x \geq 0$, $|\mathbf{f_t(x) - f_{t^1}(x)}| \leq \epsilon$ …………(2.11) for all $t , t^1 \geq T(\epsilon)$, Now let g(x) be any function in Γ^* . Hence g(x) must be in L_1 .Thus $\int_E |\mathbf{g(x)}|\mathbf{dx} < \infty$ …………(2.12) Now since we see that , $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq \int_E |\mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}}|\mathbf{dx}V \leq \epsilon \int_E |\mathbf{g(x)}|\mathbf{dx}$[By (2.11)] $\leq \boldsymbol{\epsilon}\, \mathbf{K(g)}$ [By (2.12)] for all $t , t^1 \geq T(\epsilon)$ every $\epsilon > 0$, where K(g) is a constant depending on g but independent of t in $E = [0,\infty)$ .But then by the necessary and sufficient conditions for f_t to be $\Gamma^{**}\Gamma^*$-convergent .f_t is $\Gamma^{**}\Gamma^*$-convergent .Or simply f_t is Γ^{**} Γ^*-convergent .

Theorem (2.3, II): In theorem (2.3,I) if Γ is perfect then the family f_t of functions in Γ is $\Gamma^{**}\Gamma^*$-convergent in the case f_t is parametric convergent .

Proof : Since f_t , a family of functions , is in Γ^{**} .But by hypothesis Γ is perfect Thus $\Gamma^{**} = \Gamma$ Hence f_t is in Γ^{**} implies that f_t is in Γ. Also f_t is parametric convergent in Γ^{**} so ft is parametric convergent in Γ . Hence then to every $\epsilon > 0$, there exists a positive number $T(\epsilon)$, independent of x , such that , for almost all $x \geq 0$, $|\mathbf{f_t(x) - f_{t^1}(x)}| \leq \epsilon$ ………(2.13) for all $t , t^1 \geq T(\epsilon)$, Now if g(x) be any function in Γ^*.Then g(x) must be in L_1.Hence $\int_E |\mathbf{g(x)}|\mathbf{dx} = \int_0^\infty |\mathbf{g(x)}|\mathbf{dx} < \infty$ ……..(2.14) [See Sharan , (1)] Now , since , $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq \int_E |\mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}}|\mathbf{dx} \leq \epsilon \int_E |\mathbf{g(x)}|\mathbf{dx}$ [By (2.13)] $\leq \boldsymbol{\epsilon}\, \mathbf{K(g)}$[By (2.14)] for all $t , t^1 \geq T(\epsilon)$ every $\epsilon > 0$, where K(g) is a constant depending on g but independent of t in $E = [0,\infty)$.

But then by the necessary and sufficient conditions for a family ft of functions to be projective convergent we find that f_t is $\Gamma^{**}\Gamma^*$-convergent. Or simply ft is Γ^{**} Γ^*-convergent.

Theorem (2.3, III): Let ft be a family of functions in L_∞^{**} which is parametric convergent then it is projective convergent.

Proof : Let f_t be a family of functions of x in L_∞^{**} defined for all t in $[0,\infty)$,where t is parameter , By hypothesis f_t is parametric convergent then to every $\epsilon > 0$, there exists a positive number $T(\epsilon)$, independent of x , such that , for almost all $x \geq 0$, $|\mathbf{f_t(x) - f_{t^1}(x)}| \leq \epsilon$ …….(2.15) for all $t , t^1 \geq T(\epsilon)$, Now let g(x) be any function in L_∞^* . But the dual space of L_∞ is always contained in L_1 .Hence $g(x) \in L_\infty^*$ implies g(x) must be in L_1 but then , $\int_E |\mathbf{g(x)}|\mathbf{dx} < \infty$ ……(2.16) Where $E = [0,\infty)$ .We have to show that f_t is L_∞^{**} L_∞^*-convergent ,That is to show that to every g(x) in L_∞^* and to every $\epsilon > 0$, there corresponds a positive number $T(\epsilon, g)$ such that , for all $t , t^1 \geq T(\epsilon, g)$ $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq \epsilon$ Now since , $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq |\int_E |\mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}}|\mathbf{dx} \leq \epsilon \int_E |\mathbf{g(x)}|\mathbf{dx}$ [By (2.15)] $\leq \boldsymbol{\epsilon}\, \mathbf{K(g)}$ [By (2.16)] for all $t , t^1 \geq T(\epsilon)$ every $\epsilon > 0$, where K(g) is a constant depending on g but independent of t in $E = [0,\infty)$ .Thus $f_t(x)$ is L_∞^{**} L_∞^*-convergent . Or simply without any scope of confusion ft(x) is L_∞^{**} - convergent.

However if L_∞ is perfect then $L_\infty^{**} = L_\infty$ and then the above theorem can be restated as

Theorem (2.3 , IV): Every parametric convergent family $f_t(x)$ of functions of x in L_∞ is L_∞ - convergent .

Proof : We can prove this theorem the line of proof of the just above theorem .Clearly $f_t(x)$ is in L_∞ and is parametric convergent so it does not matter that whether $f_t(x)$ is in L_∞^{**} or L_∞ as L_∞ is supposed to be perfect . Again it follows direct from the definition of parametric convergent of $f_t(x)$ in any function space α whether if $\alpha = L_\infty$ that to every $\epsilon > 0$, there corresponds a positive number $T(\epsilon)$, independent of x such that , for almost all $x \geq 0$. $|\mathbf{f_t(x) - f_{t^1}(x)}| \leq \epsilon$ ………(2.17)

For all $t , t^1 \geq T(\epsilon)$, Now in order to prove that $f_t(x)$ is L_∞ L_∞^* - convergent . We need a function g(x) in L_∞^* so as before in the previous theorem (2.3 , III) g(x) in L_∞^* implies g(x) is in L_1 because L_1 is the dual space of L_∞ . Hence $L_\infty^* = L_1$

Thus $g(x) \in L_\infty^* \Rightarrow$ g(x) is in L1 .So again we shall have that $\int_E |\mathbf{g(x)}|\mathbf{dx} < \infty$ ………… (2.18)

Now with theorem help of [(2.17) and (2.18)] we can see that $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq |\int_E |\mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}}|\mathbf{dx} \leq \epsilon \int_E |\mathbf{g(x)}|\mathbf{dx} \leq \boldsymbol{\epsilon}\, \mathbf{K(g)}$ for all $t , t^1 \geq T(\epsilon)$ every $\epsilon > 0$, where K(g) is a constant depending on g but independent of theorem parameter t in $E = [0,\infty)$ .Thus $f_t(x)$ is L_∞ L_∞^* - convergent . Or ft(x) is L_∞-convergent.

Theorem (2.3, V): In ζ^{**} parametric convergent implies ζ^{**} ζ^* - convergent.

Proof : Let $f_t(x)$ be a family of functions of x in ζ^{**} defined for all t in $[0,\infty)$, where t is a parameter . We now suppose that the family f_t is parametric convergent but then , by definition , to every $\epsilon > 0$, there exists a positive number $T(\epsilon)$, independent of x such that , for almost all $x \geq 0$.$|\mathbf{f_t(x) - f_{t^1}(x)}| \leq \epsilon$ …….(2.19) for all $t , t^1 \geq T(\epsilon)$, Let g(x) be any function in ζ^* then g(x) will get itself into theorem space of integrable functions , that is g(x) will be such that $\int_E |\mathbf{g(x)}|\mathbf{dx} < \infty$ ……(2.20) Now on the basis of [(2.19) and (2.20)] it is easy to see that $|\int_E \mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}dx}| \leq |\int_E |\mathbf{g(x)\{f_t(x) - f_{t^1}(x)\}}|\mathbf{dx} \leq \epsilon \int_E |\mathbf{g(x)}|\mathbf{dx}$ [By (2.19)] $\leq \boldsymbol{\epsilon}\, \mathbf{K(g)}$ for all $t , t^1 \geq T(\epsilon)$ every $\epsilon > 0$, where K(g) is a constant depending on g but independent of theorem parameter t in E = $[0,\infty)$ But then by a necessary and sufficient condition parametric convergent family of functions $f_t(x)$ defined for all x is $\zeta^{**}\,\zeta^{*}$ - convergent .

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