

Exploration of Hyperbolic Geometry $3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0$

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ABSTRACT

This paper aims at obtaining non-zero distinct integer solutions to the hyperbola represented by the binary quadratic equation $3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0$. A few interesting relations among the solutions are presented. Employing the solutions of the given hyperbola, solutions for other choices of hyperbolas and parabolas are presented.

Keywords: Binary quadratic, non-homogeneous quadratic, hyperbola, Integer solutions.

2010 Mathematics Subject Classification: 11D09.

I. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-18], the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by $3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

II. METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0 \quad (1)$$

Introduction of the transformations

$$x = X + 1, y = Y + 1 \quad (2)$$

in (1) gives

$$3X^2 + 3Y^2 + 7YX - 4 = 0 \quad (3)$$

Substitution of the transformations,

$$X = u + v, Y = u - v, u \neq v \neq 0 \quad (4)$$

in (3) leads to

$$v^2 = 13u^2 - 4 \quad (5)$$

whose smallest positive integer solution is $u_0 = 1, v_0 = 3$

Now, consider the pellian equation

$$v^2 = 13u^2 + 1 \quad (6)$$

whose general solution $(\tilde{u}_n, \tilde{v}_n)$ is given by

$$\tilde{v}_n = \frac{1}{2} f_n, \tilde{u}_n = \frac{1}{2\sqrt{13}} g_n$$

where $g_n = [(649 + 180\sqrt{13})^{n+1} - (649 - 180\sqrt{13})^{n+1}]$

$$f_n = [(649 + 180\sqrt{13})^{n+1} + (649 - 180\sqrt{13})^{n+1}], \quad n = 0, 1, 2, 3, \dots$$

Applying Brahmagupta lemma between the solutions (u_0, v_0) and $(\tilde{u}_n, \tilde{v}_n)$, we have

$$u_{n+1} = \frac{f_n}{2} + \frac{3g_n}{2\sqrt{13}}$$

$$v_{n+1} = \frac{3f_n}{2} + \frac{13g_n}{2\sqrt{13}}$$

Taking the advantage of (2) and (4), the sequence of integral solutions of (1) can be written as

$$13x_{n+1} = 26f_n + 8\sqrt{13}g_n + 13 \quad (7)$$

$$13y_{n+1} = -(13f_n + 5\sqrt{13}g_n) + 13 \quad (8)$$

Thus (7) and (8) represent the non-zero distinct integral solutions of (1). The above values of x_n and y_n satisfy respectively the following recurrence relations.

$$x_{n+3} = 1298x_{n+2} - x_{n+1} - 1296$$

$$y_{n+3} = 1298y_{n+2} - y_{n+1} - 1296, \quad n = 0, 1, 2, \dots$$

Some numerical examples of x and y satisfying (1) are given in the Table :1 below.

Table 1: Examples

n	x_{n+1}	y_{n+1}
-1	5	-1
0	5477	-3097
1	7107845	-4021201
2	9225976037	-5219517097

From the above table, we observe some interesting relations among the solutions which are presented below.

1. x_{n+1} values are positive and y_{n+1} values are negative. Both values are odd.

2. Each of the following expressions is a Nasty number :

- $\frac{1}{45}(-1234x_{2n+2} + x_{2n+3} + 1773)$
- $\frac{1}{58410}(-1601731x_{2n+2} + x_{2n+4} + 2302650)$
- $3(5x_{2n+2} + 8y_{2n+2} - 9)$
- $\frac{3}{611}(-5585x_{2n+2} - 8y_{2n+3} + 8037)$
- $\frac{3}{793079}(-7249325x_{2n+2} - 8y_{2n+4} + 10421649)$

3. Each of the following expressions is a Cubical integer:

- $\frac{1}{270}(-1234x_{3n+3} + x_{3n+4} - 3702x_{n+1} + 3x_{n+2} + 4932)$
- $\frac{1}{350460}(-1601731x_{3n+3} + x_{3n+5} - 4805193x_{n+1} + 3x_{n+3} + 6406920)$
- $\frac{1}{2}(5x_{3n+3} + 8y_{3n+3} + 15x_{n+1} + 24y_{n+1} - 52)$
- $\frac{1}{1222}(-5585x_{3n+3} - 8y_{3n+4} - 16755x_{n+1} - 24y_{n+2} + 22372)$
- $\frac{1}{1586158}(-7249325x_{3n+3} - 8y_{3n+5} - 21747975x_{n+1} - 24y_{n+3} + 28997332)$

4. Each of the following expressions is a Bi-Quadratic integer:

- $\frac{1}{270}(-1234x_{4n+4} + x_{4n+5} - 4936x_{2n+2} + 4x_{2n+3} + 7785)$

- $\frac{1}{350460}(-1601731x_{4n+4} + x_{4n+6} - 6406924x_{2n+2} + 4x_{2n+4} + 10111410)$
- $\frac{1}{2}(5x_{4n+4} + 8y_{4n+4} + 20x_{2n+2} + 32y_{2n+2} - 53)$
- $\frac{1}{1222}(-5585x_{4n+4} - 8y_{4n+5} - 22340x_{2n+2} - 32y_{2n+3} + 35297)$
- $\frac{1}{1586158}(-7249325x_{4n+4} - 8y_{4n+6} - 28997300x_{2n+2} - 32y_{2n+4} + 45763613)$

5. Each of the following expressions is a Quintic integer:

- $\frac{1}{270}(-1234x_{5n+5} + x_{5n+6} - 6170x_{3n+3} + 5x_{3n+4} - 12340x_{n+1} + 10x_{n+2} + 19728)$
- $\frac{1}{350460}\left(-1601731x_{5n+5} + x_{5n+7} - 8008655x_{3n+3} + 5x_{3n+5} - 16017310x_{n+1}\right. \\ \left.+ 10x_{n+3} + 25627680\right)$
- $\frac{1}{2}(5x_{5n+5} + 8y_{5n+5} + 25x_{3n+3} + 40y_{3n+3} + 50x_{n+1} + 80y_{n+1} - 208)$
- $\frac{1}{1222}(-5585x_{5n+5} - 8y_{5n+6} - 27925x_{3n+3} - 40y_{3n+4} - 55850x_{n+1} - 80y_{n+2} + 89488)$
- $\frac{1}{1586158}\left(-7249325x_{5n+5} - 8y_{5n+7} - 36246625x_{3n+3} - 40y_{3n+5}\right. \\ \left.- 72493250x_{n+1} - 80y_{n+3} + 188482658\right)$

6. Relations among the solutions:

- $13x_{n+3} = -13x_{n+1} + 16874x_{n+2} - 16848$
- $1298x_{n+2} = x_{n+1} + x_{n+3} + 1296$
- $319337750160y_{n+3} = -228000x_{n+1} - 180662603121x_{n+3} + 500000581160$
- $1222y_{n+3} = 2160x_{n+1} + 1586158y_{n+2} - 1587096$
- $793079x_{n+3} = -x_{n+1} - 1401840y_{n+3} + 2194920$
- $793079y_{n+2} = -1080x_{n+1} + 611y_{n+3} + 793548$

Remarkable Observations

(i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2 below.

Table 2: Hyperbolas

S.NO	Hyperbolas	(X,Y)
1	$208X^2 - Y^2 = 60652800$	$\left(-1234x_{n+1} + x_{n+2} + 1233, \right. \\ \left.17797x_{n+1} - 13x_{n+2} - 17784\right)$
2	$208X^2 - Y^2 = 102188080051200$	$\left(-1601731x_{n+1} + x_{n+3} + 1601730, \right. \\ \left.23100493x_{n+1} - 13x_{n+3} - 23100480\right)$
3	$13X^2 - Y^2 = 208$	$\left(5x_{n+1} + 8y_{n+1} - 13, \right. \\ \left.-13x_{n+1} - 26y_{n+1} + 39\right)$
4	$X^2 - 13Y^2 = 5973136$	$\left(-5585x_{n+1} - 8y_{n+2} + 5593, \right. \\ \left.1549x_{n+1} + 2y_{n+2} - 1551\right)$
5	$13X^2 - Y^2 = 130826654450128$	$\left(-7249325x_{n+1} - 8y_{n+3} + 7249333, \right. \\ \left.26137813x_{n+1} + 26y_{n+3} - 26137839\right)$

(ii) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3 below.

Table3: Parabolas

S.NO	Parabolas	(X,Y)
1	$56160 X - Y^2 = 60652800$	$\begin{pmatrix} -1234x_{2n+2} + x_{2n+3} + 1773, \\ 17797x_{n+1} - 13x_{n+2} - 17784 \end{pmatrix}$
2	$72895680 X - Y^2 = 1021880800 \ 51200$	$\begin{pmatrix} -1601731x_{2n+2} + x_{2n+4} + 2302650, \\ 23100493x_{n+1} - 13x_{n+3} - 23100480 \end{pmatrix}$
3	$26X - Y^2 = 208$	$\begin{pmatrix} 5x_{2n+2} + 8y_{2n+2} - 9, \\ -13x_{n+1} - 26y_{n+1} + 39 \end{pmatrix}$
4	$94X - Y^2 = 459472$	$\begin{pmatrix} -5585x_{2n+2} - 8y_{2n+3} + 8037, \\ 1549x_{n+1} + 2y_{n+2} - 1551 \end{pmatrix}$
5	$20620054 X - Y^2 = 1308266544 \ 50128$	$\begin{pmatrix} -7249325x_{2n+2} - 8y_{2n+4} + 10421649, \\ 26137813x_{n+1} + 26y_{n+3} - 26137839 \end{pmatrix}$

III. CONCLUSION

In conclusion, one may search for other patterns of solutions and their corresponding properties.

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